# STURM COMPARISON THEOREMS FOR ELLIPTIC INEQUALITIES

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Communicated by M. H. Protter, May 21, 1969

Comparison theorems of Sturm's type will be stated for the quasilinear elliptic partial differential inequalities

$$(1)\quad lu = -\sum_{i,j=1}^{n} D_i[a_{ij}(x, u)D_ju] + 2\sum_{i=1}^{n} b_i(x, u)D_iu + uc(x, u) \leqq 0,$$

$$(2)\quad Lv = -\sum_{i,j=1}^{n} D_i[A_{ij}(x, v)D_jv] + 2\sum_{i=1}^{n} B_i(x, v)D_iv + vC(x, v) \geqq 0,$$

$$x = (x_1, \cdots, x_n) \in G,\quad u, v \in I,\quad D_i = \partial/\partial x_i\quad (i = 1, \cdots, n)$$

where $G$ is a nonempty regular bounded domain in $R^n$ and $I$ is a real interval containing zero. The functions $a_{ij}$, $A_{ij}$, $b_i$, $B_i$, $c$, and $C$ are assumed to be real-valued and continuous on $\bar{G}\times I$, and the matrices $(a_{ij})$ and $(A_{ij})$ symmetric and positive definite in $G\times I$.

A *Sturmian theorem* has the following form: If (1) has a nontrivial solution $u$ which vanishes identically on the boundary of $G$ and if (2) majorizes (1) in some sense, then every solution $v$ of (2) has a zero in $\bar{G}$ (or $G$).

The linear selfadjoint case ($b_i = B_i = 0$, $i = 1, \cdots, n$) was first considered by Picone [12], and later independently and more generally by Hartman and Wintner [4], Kuks [10], Kreith [6], [8], Clark and Swanson [2]. A recent research announcement of Diaz and McLaughlin [3] is similar to Kreith's "strong comparison theorem" [9], obtained when $\partial G$ has the "sphere property" by an appeal to the Hopf maximum principle. The conclusion of the strong comparison theorem is that $v$ has a zero in $G$ unless $v$ is a constant multiple of $u$; an analogous result in the quasilinear case is stated below (Theorem 2). Earlier McNabb [11] had used similar techniques in a different connection.

The linear nonselfadjoint case was studied by Protter [13], Swanson [16], Kreith [9], and Allegretto [1]. Extensions to unbounded domains were obtained in [16] and [17] and applied to oscillation theory and eigenvalue estimation. Comparison theorems

[1] Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant No. AFOSR-68-1531.