# ISOMETRIC EMBEDDINGS[1]

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This note states results extending those of Nash [2] on isometric embeddings of Riemannian manifolds in euclidean spaces; proofs and further details will be given elsewhere.

Let $M$ be a $d$-dimensional $C^\infty$ manifold. For convenience, we assume throughout that manifolds, whether compact or not, are connected. A *metric* on $M$ is defined to be a quadratic form on the tangent bundle of $M$; *note that there is no assumption of nondegeneracy.* We shall assume that all metrics are $C^\infty$. A *Riemannian metric* on $M$ is a metric whose restriction to the tangent space $T_q$ at a point $q \in M$ is positive definite, for all $q \in M$. A *pseudo-Riemannian*, or *indefinite*, *metric* is a metric whose restriction to the tangent space at each point is nondegenerate; if the nondegenerate restriction to $T_q$ has $n$ negative eigenvalues and $p$ positive eigenvalues, with $p+n=d$, the metric is said to have signature $(p, n)$ at $q$. The connectedness of $M$ implies that the signature is independent of the choice of $q \in M$.

$R^m$ will denote euclidean $m$-dimensional space, with the standard flat, positive definite metric, unless otherwise indicated; $R_n^p$ denotes euclidean $(n+p)$-dimensional space with flat metric of signature $(p, n)$. Thus $R_0^m = R^m$. Let $F$ be a $C^\infty$ map, $F: M \to R_n^p$, and let $g$ be a metric on $M$; $F$ is said to be isometric for $g$ if $F^*(\cdot) = g$ where "$\cdot$" denotes the metric for $R_n^p$ indicated above. Note that if $g$ is Riemannian and $F$ is isometric for $g$, then $F$ is necessarily an immersion and $n+p \geqq d = \dim M$; for a general metric $g$, however, $F$ need not be an immersion. We shall concern ourselves with the question: given $M$ and a metric $g$ on $M$, for what $R_n^p$ do there exist isometric immersions, or isometric embeddings, $F: M \to R_n^p$?

**1. A geometric argument for general metrics.** Nash [2] guarantees the existence of isometric embeddings in some Riemannian euclidean space for any manifold with a Riemannian metric. The following argument reduces the general metric case to the Riemannian case, but requires higher dimension in the receiving euclidean space than necessary.

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