TWO L^p INEQUALITIES[1]

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We present here two new inequalities for the space of vector-valued functions X in L^p, $p>1$ with the norm $\|X\|$ satisfying $\|X\|^p = \int |X|^p d\mu$. The inequalities are extensions of those given by K. O. Friedrichs [1] and can be used respectively instead of Clarkson's inequality, [2], to give simple proofs that L^p space is uniformly convex (rotund) and uniformly smooth. A different proof of the uniform convexity was given by Beurling in a lecture and for $p \geqq 2$ by Mostow [3]. For earlier results on the uniform smoothness see Day [4].

The two inequalities (global) are for $p>1$,

$$\text{(Ia)}\qquad \frac{\|X\|^p + \|Y\|^p}{2} - \left\|\frac{X+Y}{2}\right\|^p \geqq a\left\|\frac{X-Y}{2}\right\|^{p/s}\left(\frac{\|X\|^p + \|Y\|^p}{2}\right)^{1-(1/s)}$$

where $a=a(p)>1$, $s=1$ for $p<2$, $s=p/2$ for $p>2$, and

$$\text{(Ib)}\qquad (\tfrac{1}{2}\|X\|^p + \tfrac{1}{2}\|Y\|^p) \leqq \left\|\frac{X+Y}{2}\right\|^p\left(1 + b_1\left(\frac{\|X-Y\|}{\|X+Y\|}\right)^s + b_2\left(\frac{\|X-Y\|}{\|X+Y\|}\right)^p\right)$$

where $b_1=b_1(p)$, $s=2$, vanishes for $p \leqq 2$ and $b_2=b_2(p)$. Note by convexity since $p>1$,

$$\text{(1)}\qquad \left\|\frac{X+Y}{2}\right\| \leqq \tfrac{1}{2}(\|X\| + \|Y\|) \leqq (\tfrac{1}{2}\|X\|^p + \tfrac{1}{2}\|Y\|^p)^{1/p} \leqq \max(\|X\|, \|Y\|).$$

We set $X+Y=2A$, $X-Y=2D$ and introduce $r=\|D\|/\|A\|$ and $m=(\frac{1}{2}\|X\|^p+\frac{1}{2}\|Y\|^p)^{1/p}$. Then one notes that the two inequalities may be used to confine the ratio $\|A\|/m$ in the form

$$(1 + b_1 r^2 + b_2 r^p)^{1/p} \leqq \|A\|/m \leqq (1 - (cr\|A\|/m)^{p/s})^{1/p}$$

where $c=c(p)<1$.

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