MARKOV PROPERTIES OF BROWNIAN LOCAL TIME

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1. Starred references are to Itô and McKean [1], the terminology and notation of which are used here. Our aim is to indicate an elementary approach to local time theory which highlights certain Markov properties discovered by Ray and Knight (§2.8*). Details and further applications of our method will be given elsewhere.

Let $X=\{x(t):t\geqq 0\}$ be standard Brownian motion and, for $t\geqq 0$, set

$$\phi(t) = \text{measure}\{s: 0 \leqq s \leqq t;\, x(s) \geqq 0\},$$
$$\rho(t) = \inf\{s: \phi(s) > t\}, \qquad y(t) = x(\rho(t)).$$

Then $Y=\{y(t):t\geqq 0\}$ is a reflecting Brownian motion (§2.11*) with local time

$$L(t) = \lim_{\epsilon\downarrow 0} (2\epsilon)^{-1}\,\text{measure}\{s: 0 \leqq s \leqq t,\, 0 \leqq y(s) < \epsilon\}.$$

THEOREM 1. *For $x\geqq 0$, $t\geqq 0$ and $\theta>0$,*

$$E_x\{\exp[-\theta\rho(t)] \mid Y\} = \exp\,[-\theta t - (2\theta)^{1/2}L(t)]. \tag{1}$$

Notes. Lévy's arc-sine law (§2.6*, Equation (17)) follows quickly on taking expectations of both sides of Equation (1). Excursion theory (§2.9*) renders Theorems 1 and 2 intuitively clear.

PROOF. Introduce an exponential variable ζ (with rate θ) independent of X. Set

$$\phi^*(t) = \phi(t\wedge\zeta), \qquad \rho^*(t) = \inf\{s: \phi^*(s) > t\}, \qquad y^*(t) = x(\rho^*(t)).$$

Then $Y^*=\{y^*(t):t<\phi(\zeta)\}$ is a diffusion on $[0,\infty)$ with generator

$$G^*f = \tfrac{1}{2}f'' - \theta f \quad \text{on} \quad C^2[0,\infty) \cap \{f: f^+(0) = \gamma f(0)\},$$

where $\gamma=(2\theta)^{1/2}$. Thus the transition function P^* of Y^*, which induces corresponding measures

$$P_x^* \ (x \geqq 0) \quad \text{on} \quad B[Y] = B[y(s): s \geqq 0],$$

is of the form

$$P^*(t) = \exp(-\theta t)P^{\cdot}(t)$$

[1] This work was done at the University of Cambridge.