# COMMUTING VECTORFIELDS ON OPEN MANIFOLDS[1]

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Let $M$ be an open orientable differentiable $n$-manifold. More precisely, we will take $M$ and vectorfields over $M$ to be of class $C^\infty$. A nonzero vectorfield $X$ on $M$ will be called *nonrecurrent* if the 1-dimensional foliation associated with $X$ is regular (see [4, Chapter I]) and admits no compact leaves. The notation $H^p(M; Z)=Q$ shall mean that the $p$-dimensional singular integral cohomology of $M$ is trivial or admits no torsion of order 2, depending on whether $p$ is even or odd, respectively.

THEOREM 1. *Let $X$ be a nonrecurrent vectorfield on $M$ and let $A\subset M$ be relatively compact. When $H^{n-1}(M; Z)=Q$ there exists a vectorfield $Y$ on $A$ such that $X$, $Y$ are linearly independent and commute.*

THEOREM 2. *When $H^{n-1}(M; Z)=Q$ every relatively compact subset of $M$ submerges in the plane.*

For $n>4$ Theorem 2 is implied by a result of I. M. James and E. Thomas (quoted as Theorem 8.6 in [5]). Moreover, we note that the cohomological triviality condition is crucial to both Theorems 1 and 2. A very simple example shows this in the case of Theorem 1: Let $M$ be Euclidean 3-space with a point 0 removed and let $X=\partial/\partial r$, where $r$ denotes distance to 0. Let $S$ denote the unit sphere centered at 0 and let $\pi: M\to S$ denote radial projection. There exist relatively compact subsets $A\subset M$ such that $\pi(A)=S$. A vectorfield $Y$ on $A$ which commutes with $X$ induces then a vectorfield $\bar{Y}$ on $S$ such that $\bar{Y}$ pulls back to $Y$ under $d\pi$. Moreover, if $(X, Y)$ are linearly independent, $\bar{Y}$ must be nonzero, showing that the conclusion of Theorem 1 does not hold in this case. It is also possible to display examples of open orientable $C^\infty$-manifolds $M$ with relatively compact $A\subset M$ which do not submerge in the plane. We may take $M$ to be the punctured real projective space of dimension 5, for instance. It is known [5, p. 201] that this space does not submerge in the plane. But obviously $M$ admits relatively compact subsets $A$ which are in fact diffeomorphic to $M$.

In this note we shall derive Theorems 1 and 2 from results established in [6]. First a few definitions: If $F$ is a regular orientable $p$-

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