MANIFOLDS OF THE HOMOTOPY TYPE OF (NON-LIE) GROUPS

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Hilbert's Fifth Problem implies that a topological group which is topologically a finite dimensional manifold is a Lie group. Until quite recently, the only topological groups of the homotopy type of compact manifolds known were Lie groups. In 1963 Slifker exhibited a topological group of the homotopy type of S^3 yet not multiplicatively equivalent to SU(1). In 1968, Hilton and Roitberg announced the discovery of a 10-dimensional manifold M_7^{10} which admits a multiplication yet is not of the homotopy type of a Lie group. In fact, they showed $M_7^{10}\times S^3=\mathrm{Sp}\,(2)\times S^3$. They left open the question: Does M_7^{10} admit a homotopy associative multiplication, a necessary condition for M^{10} to be of the homotopy type of a topological group? We answer the question affirmatively; thus a homotopy version of Hilbert's Fifth Problem is false.

THEOREM 1. *There is a topological group G of the homotopy type of a compact manifold M^{10} (the 3-sphere bundle over S^7 described by Hilton and Roitberg) which is not of the homotopy type of any Lie group.*

More precisely we show the following

THEOREM 2. *Let $S^3\to M_n^{10}\to S^7$ be the principal S^3-bundle classified by $n\omega\in\pi_6(S^3)$, $n\in Z_{12}$, ω chosen as a generator such that the corresponding M_1^{10} is* Sp (2).

M_n^{10} is of the homotopy type of a Lie group if and only if $n\equiv\pm 1$ (12).

M_n^{10} is of the homotopy type of a topological group if $n\equiv\pm1,\ \pm5$ (12).

M_n^{10} admits a multiplication if $n\not\equiv 2$ (4).

The first part results from the classification of such bundles up to homotopy type and the classification of Lie groups. The case $n\equiv -1$ is realized by $\overline{\mathrm{Sp}}\,(2)$, the opposite symplectic group, which has the same underlying space as Sp (2) but the opposite order of multiplication.

The remainder of the theorem is proved using a new technique of Zabrodsky's called "mixing homotopy types" [**2**].

Let P be the set of primes and $P=P_1\cup P_2$, a decomposition into disjoint subsets. Let $\mathcal{C}P_1$ denote the class of abelian groups of orders not divisible by primes in P_2 and let $\mathcal{C}P_2$ denote the class of abelian groups not divisible by primes in P_1.

Let X, X_0 be simply connected CW-complexes.