ON THE DECOMPOSITION OF MODULES

BY ANDREAS DRESS

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Let R be a commutative ring with $1 \in R$, A and R-algebra—not necessarily commutative—and let M, N be two A-left-modules. We write $N-\mathrm{rk}(M) \geqq s$, if $M \cong sN \oplus M'$ for some A-left-module M' with $s \cdot N$ short for $N \oplus N \oplus \cdots \oplus N$, s-times.

Then one can prove the following generalization of a theorem of Serre (cf. [1] or [4]).

THEOREM 1. *Assumptions.*

(i) *N is finitely presented as A-left-module,* $\mathrm{End}_A(N)$ *finitely generated as R-module and M a direct summand in a direct sum of finitely presented A-modules*;

(ii) *the maximal ideal spectrum of R is noetherian of dimension d*;

(iii) *for any maximal ideal $\mathfrak{m}$ in R we have $N_{\mathfrak{m}} - \mathrm{rk}(M_{\mathfrak{m}}) \geqq d+s$ with $N_{\mathfrak{m}}$, resp. $M_{\mathfrak{m}}$ the $A_{\mathfrak{m}} = R_{\mathfrak{m}} \otimes_R A$-module $R_{\mathfrak{m}} \otimes_R N$, resp. $R_{\mathfrak{m}} \otimes M$.*

Then $N-\mathrm{rk}(M) \geqq s$.

Moreover, if R is noetherian, $\hat{R}_{\mathfrak{m}}$ the $\mathfrak{m}$-adic completion of R for some maximal ideal $\mathfrak{m}$ and $\hat{N}_{\mathfrak{m}}$, resp. $\hat{M}_{\mathfrak{m}}$ the $\hat{A}_{\mathfrak{m}} = \hat{R}_{\mathfrak{m}} \otimes_R A$-module $\hat{R}_{\mathfrak{m}} \otimes_R N$, resp. $\hat{R}_{\mathfrak{m}} \otimes_R M$, then

$$N_{\mathfrak{m}} - \mathrm{rk}(M_{\mathfrak{m}}) \geqq d + s \Leftrightarrow \hat{N}_{\mathfrak{m}} - \mathrm{rk}(\hat{M}_{\mathfrak{m}}) \geqq d + s.$$

One can also prove the following generalization of the Cancellation Theorem of Bass (cf. [1]).

THEOREM 2. *Assumptions.*

(i) *and* (ii) *as in Theorem* 1;

(iii) *M contains a direct summand P with $N-\mathrm{rk}_{\mathfrak{m}}(P) > d$ for all maximal ideals $\mathfrak{m}$ in R, which is a direct summand in some $s \cdot N$*;

(iv) *Q is an A-left-module, which is also a direct summand in some $s \cdot N$, and M' is some A-left-module with $Q \oplus M \cong Q \oplus M'$.*

Then $M \cong M'$.

The proof follows closely those of Serre and Bass [1], [4], once the following observations have been made:

(1) If N is any A-left-module and if $B = \mathrm{End}_A(N)$—acting from the right on N—then the contravariant functor $\mathrm{Hom}_A(\cdot, N)$ from A-left-modules to B-right-modules defines a contravariant equivalence between the category $[N]$ of those A-left-modules P, which are a direct summand in some $s \cdot N$ (and all possible A-homomorphisms as morph-