A THEOREM OF STOUT

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In a paper that appeared in Mathematische Annalen in 1968, E. L. Stout proved a generalization of a theorem of Rado which states that if f is continuous on the set $D = \{z: |z| < 1\}$ and holomorphic on $D - f^{-1}(0)$, then f is holomorphic on D. Stout's generalization is as follows:

THEOREM. *Let E be a set of capacity zero in the complex plane, and let E_0 be a relatively closed set in D. If f is a function bounded and holomorphic in $D - E_0$, if f' does not vanish identically, and if for every sequence $\{z_n\}$ in $D - E_0$ such that $z_n \to z_0 \in E_0$ and $\lim f(z_n) = w_0$ exists it is the case that $w_0 \in E$, then f is holomorphic throughout D.*

We shall generalize this theorem by replacing D by an arbitrary hyperbolic Riemann surface and replacing the assumption that f is bounded by the assumption that f belong to the Hardy class H^P, $0 < P \leq \infty$, i.e., that $|f|^P$ possesses a harmonic majorant. Thus our theorem reads as follows:

THEOREM. *Let E be a set of capacity zero in the complex plane, and let E_0 be a closed subset of a hyperbolic Riemann surface R. If $f \in H^P(R - E_0)$, i.e., f is of class H^P on each component of $R - E_0$, $0 < P \leq \infty$, if f is nonconstant on some component of $R - E_0$, and if for every sequence $\{a_n\}$ in $R - E_0$ such that $a_n \to a_0 \in E_0$ and $\lim f(a_n) = b_0$ exists it is the case that $b_0 \in E$, then f is holomorphic throughout D.*

REMARK. In the proof we shall use the same notation and terminology in [1] and [2].

PROOF. Let $S = R - E_0$ and let S_0 be a component of S on which f is nonconstant. It then follows that S_0 is hyperbolic and that its universal covering surface is D. Let π denote the projection map of D onto S_0. Since π is a Fatou mapping, the fine limit function $\hat{\pi}$ is defined a.e. on ∂D. Here a.e. refers to Lebesgue measure. Since D is a regular covering surface of S_0, it follows that every point P' in S_0 has a neighborhood V with the property that each component of $\pi^{-1}(V)$ is compact. Hence π is of Blaschke type. It follows that for a.e. point

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