ON SIMULTANEOUS APPROXIMATION AND INTERPOLATION WHICH PRESERVES THE NORM

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Communicated by Avner Friedman, February 14, 1969

In [**6**] H. Yamabe established the following "simultaneous approximation and interpolation" theorem, which generalized a result of Walsh [**4**, p. 310] (cf. also [**1**], [**3**] for further generalizations), and is related to a theorem of Helly in the theory of moments (cf. e.g. [**2**, pp. 86–87]).

THEOREM (YAMABE). *Let M be a dense convex subset of the real normed linear space X, and let $x_1^*, \cdots, x_n^* \in X^*$. Then for each $x \in X$ and each $\epsilon > 0$, there exists a $y \in M$ such that $\|x-y\| < \epsilon$ and $x_i^*(y) = x_i^*(x)$ $(i=1, \cdots, n)$.*

Wolibner [**5**], in essence, proved that Yamabe's theorem could be sharpened in the particular case when $X = C([a, b])$, $M = \mathcal{P} =$ "the polynomials," and the x_i^* are "point evaluations." Indeed, from the results of [**5**] there can readily be deduced the following

THEOREM (WOLIBNER). *Let $a \leqq t_1 < \cdots < t_n \leqq b$ and let $\mathcal{P}$ be the set of polynomials. Then for each $x \in C([a, b])$ and each $\epsilon > 0$, there exists a $p \in \mathcal{P}$ such that $\|x-p\| < \epsilon$, $p(t_i) = x(t_i)$ $(i=1, \cdots, n)$, and $\|p\| = \|x\|$.*

Motivated by Wolibner's theorem, we consider the following more general problem. Let M be a dense subspace of the real normed linear space X, and let $\{x_1^*, \cdots, x_n^*\}$ be a finite subset of the dual space X^*. The triple $(X, M, \{x_1^*, \cdots, x_n^*\})$ will be said to have *property SAIN* (simultaneous approximation and interpolation which is norm-preserving) provided that the following condition is satisfied:

For each $x \in X$ and each $\epsilon > 0$ there exists a $y \in M$ such that $\|x-y\| < \epsilon$, $x_i^*(y) = x_i^*(x)$ $(i=1, \cdots, n)$, and $\|y\| = \|x\|$.

In this note we shall outline some of the main results we have obtained regarding property SAIN. Detailed proofs and related matter will appear elsewhere.

THEOREM 1. *Let M be a dense subspace of the Hilbert space X and let $x_1^*, \cdots, x_n^* \in X^*$. Then $(X, M, \{x_1^*, \cdots, x_n^*\})$ has property SAIN if and only if each x_i^* attains its norm on the unit ball in M.*

[1] Supported by grants from the National Science Foundation.