CONVEX COMBINATIONS OF UNIMODULAR FUNCTIONS

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Stephen Fisher has recently proved [1] that the set of all convex combinations of finite Blaschke products is dense in the unit ball of the disc algebra. The restriction of the disc algebra to the unit circle T is the subalgebra A of $C(T)$ which consists of those $f \in C(T)$ whose Fourier transforms are supported by the semigroup of the nonnegative integers. The unimodular members of A (i.e., those $f \in A$ for which $|f| = 1$ on T) are the (restrictions to T of the) finite Blaschke products. Hence Fisher's result is a special case of the following

THEOREM. *Let Γ be the dual of a compact abelian group G, let Σ be a semigroup in Γ, let C_Σ consist of all $f \in C(G)$ whose Fourier transforms $\hat{f}$ are supported by Σ. Then the set of all finite convex combinations of unimodular members of C_Σ is dense (relative to the supremum norm) in the unit ball of C_Σ.*

The proof is based on the three lemmas stated below. Lemmas 1 and 2 are of some interest even in the classical case $G = T$. Lemma 3 is a technicality; it is used in the proof of Lemma 2. Terminology and notation are as in [3]. In particular, the $*$ in Lemma 1 denotes convolution, and $M(G)$ is the set of all complex Borel measures on G.

LEMMA 1. *Suppose Q is a closed, convex, balanced subset of $C(G)$ which is translation-invariant. Suppose $f \in C(G)$ but $f \notin Q$. Then there exists $\mu \in M(G)$ such that*

(i) $(f * \mu)(0) > 1$,

(ii) $\|g * \mu\|_\infty < 1$ *for every* $g \in Q$,

(iii) $\hat{\mu}$ *has finite support in* Γ.

LEMMA 2. *Suppose Σ is a semigroup in Γ, Σ is not a group, $f \in C_\Sigma$, $\|f\|_\infty < 1$, and E is a finite subset of Γ. Then there exists a unimodular $g \in C_\Sigma$ such that $\hat{g}(\gamma) = \hat{f}(\gamma)$ for every $\gamma \in E$.*

LEMMA 3. *If Λ is a finitely generated abelian group and if S is a semigroup in Λ which is not a group, then there is a homomorphism ϕ of Λ into the real line such that $\phi(s) \geqq 0$ for every $s \in S$ and $\phi(s_0) = 1$ for some $s_0 \in S$.*

The theorem is an easy consequence of Lemmas 1 and 2. First, if Σ is a group, then C_Σ is the same as $C(H)$ where H is the quotient of

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