ON SUBALGEBRAS OF C^*-ALGEBRAS

BY WILLIAM B. ARVESON[1]

Communicated by Murray Gerstenhaber, January 14, 1969

In this note we announce some new methods and results in the theory of nonnormal Hilbert space operators and nonselfadjoint operator algebras. A main difficulty in the subject has been the apparent absence of relations between, say, a nonselfadjoint algebra of operators and its generated C^*-algebra. For example, given full information about the norm-closed algebra $P(T)$ generated by all polynomials in a given (nonnormal) operator T, what can one say about the C^*-algebra $C^*(T)$ generated by T and the identity? While one cannot expect much of an answer in general, we will describe here a class of operators and operator algebras for which these relations are as simple as one could hope for.

All C^*-algebras are assumed to contain an identity (written as e), $L(H)$ denotes the algebra of all bounded operators on a Hilbert space H, and $C^*(S)$ stands for the C^*-algebra generated by S and the identity where S is either an operator or a subset of a C^*-algebra. An operator is irreducible if it commutes with no nontrivial projections.

1. **An extension theorem.** Let S be a linear subspace of a C^*-algebra B, such that S contains the identity of B. A linear map ϕ of S into another C^*-algebra is *positive* if $\phi(x) \geqq 0$ for every positive element x of S (note, however, that S may contain no positive elements other than scalars). A familiar theorem of M. Krein implies that if $S = S^*$, then every scalar-valued positive linear map of S has a positive extension to B. We first describe a generalization of Krein's theorem to operator-valued maps which is basic for virtually all of the sequel. If M_n is the algebra of all complex $n \times n$ matrices, then $B \otimes M_n$ is the $*$-algebra of all $n \times n$ matrices over B. There is a unique C^*-algebra norm on $B \otimes M_n$, and $S \otimes M_n$ is a linear subspace of this C^*-algebra. A linear map ϕ of S into a C^*-algebra B' induces, for every $n \geqq 1$, a linear map $\phi_n : S \otimes M_n \to B' \otimes M_n$ by applying ϕ element by element to each matrix over S. ϕ is *completely contractive* or *completely isometric* according as each ϕ_n is contractive ($\|\phi_n\| \leqq 1$) or isometric. ϕ is *completely positive* if each ϕ_n is a positive linear map.

[1] Research supported by NSF grant GP-5585 and the U.S. Army Research Office, Durham.