# TRIANGULATION OF MANIFOLDS. II

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**Main theorems.** We will say that a manifold $M$ satisfies condition S, if $\pi_1(M\times T^k)$ and $\pi_1(\partial M\times T^k)$ satisfy the conditions necessary for the splitting theorem to hold [6], [9].

THEOREM 4. *Every closed topological manifold $M$, dim $M\geqq 5$, $H^4(M;\ Z_2)=0$, and satisfying condition $S$, admits a PL manifold structure.*[2]

PROOF. By Theorem 3 and addendum to Theorem 2, the tangent bundle of $M^n$ lifts to a $\mathrm{PL}_n$-bundle. By the splitting theorem [6], [9], there is a PL-manifold $Q$ of the same tangential homotopy type as $M$. As in [5], proof of (c), we may immerse $M_0=M$-point in $Q$, to give $M_0$ a PL manifold structure. By Lees' Lemma [5], $M$ admits a PL manifold structure.

REMARKS. 1. If we are given a lift of $\tau(M^n)$ to a $\mathrm{PL}_n$-bundle, we may drop the condition $H^4(M;\ Z_2)=0$.

2. If we are given a bundle map of $\tau(M_0)$ into $\tau(Q)$, $Q^n$ a PL manifold, we may drop condition S as well.

THEOREM 5. *Let $W^n$, $n\geqq 5$, be a topological h-cobordism between PL manifolds. If $H^3(W;\ Z_2)=0$, then $W$ admits a PL manifold structure with the given structures on the boundary.*

PROOF. Say $\partial W=M_1\cup M_2$. Then we may define inclusions $\iota_1$: $M_1\times I\rightarrow W$, $\iota_2$: $M_2\times I\rightarrow W$ using collar neighborhoods. (Take $\iota_1|M_1\times 0=\text{identity}$ and $\iota_2|M_2\times 1=\text{identity}$.) Also we have retractions $r_1$: $W\rightarrow M_1\times I$, $r_2$: $W\rightarrow M_2\times I$, where for example we may take $r_2|M_2$: $M_2\rightarrow M_2\times 1$ by the identity, $r_2|M_1$: $M_1\rightarrow M_2\times 0$ by a homotopy equivalence, and $r_2\iota_2=\text{identity}$. Now these maps are covered by topological bundle maps; $\iota_i^*$: $\tau_1\oplus 1\rightarrow\tau=\tau(W)$, $\iota_2^*$: $\tau_2\oplus 1\rightarrow\tau$, and $r_2^*$: $\tau\rightarrow\tau_2\oplus 1$ so that $r_2^*\iota_2^*=\text{identity}$ (since $M_2\times I$ is a deformation retract of $W$). Then $r_2^*\iota_1^*$: $\tau_1\oplus 1\rightarrow\tau_2\oplus 1$ is a topological bundle map.

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[2] As first shown by Kirby and Siebenmann (by other methods), condition $S$ may be eliminated. We can do this by applying Theorem 7 below to the normal disk bundle of a compact manifold $M$ (condition 3 is unnecessary since the tangent bundle is trivial) to obtain their result that $M$ is the homotopy type of a finite complex. The splitting theorem then holds with no condition on the fundamental group [9].