ON THE EXISTENCE AND IRREDUCIBILITY OF CERTAIN SERIES OF REPRESENTATIONS[1]

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1. Introduction. 1. By the principal series one means here the unitary representations of a semisimple Lie group G arising from induction to G by characters on MAN corresponding to characters on A. Although long conjectured to be irreducible, this family of representations has been shown to be irreducible only for special groups. For example see [9] for complex G and see [3] for the group $\mathrm{Sl}(n, \boldsymbol{R})$. In the general case (all G) irreducibility has been proved by Bruhat [1] using analytic methods, only however, for the "regular" characters on A. A proof of the irreducibility of all the elements of the principal series is but one application of certain algebraic results, stated here, on modules of the universal enveloping algebra U of the Lie algebra $\mathfrak{g}$ of G.

A second application is the proof of the existence and irreducibility of the complementary series for all semisimple Lie groups generalizing in a natural way the case of $\mathrm{Sl}(2, \boldsymbol{R})$. It is shown also that if $\dim A = 1$ (split rank 1 case) then except for possibly the trivial (one dimensional) representation the most general irreducible unitary representation of G admitting a fixed vector for K ($\mathrm{Ad}_{\mathfrak{g}}K$ is the maximal compact subgroup of $\mathrm{Ad}_{\mathfrak{g}}G$) belongs either to the principal or complementary series.

1.2. If $\mathfrak{a}_{\mathbf{C}}'$ is the complex dual to the Lie algebra a of A then any $\lambda \in \mathfrak{a}_{\mathbf{C}}'$ defines a one dimensional representation $b \to b^{\lambda}$ of $B = MAN$. If X^{λ} is the space of all analytic K-finite functions f on G such that $f(ab) = b^{-\lambda} f(a)$ where $a \in G$, $b \in B$ then X^{λ} is in a natural way a U-module. The results above are mainly applications of a theorem (Theorem 2) giving a necessary and sufficient condition on λ for X^{λ} to be an irreducible (in the usual algebraic sense) U-module. In particular there arises, in a natural way, a region in $\mathfrak{a}_{\mathbf{C}}'$ which we call the critical strip (CS) for which X^{λ} is always U-irreducible. The critical strip contains all of the λ corresponding to the principal series and its closure contains all the λ corresponding to the complementary series.

[1] By invitation the author addressed the annual meeting of the American Mathematical Society in Cincinnati on January 22, 1962 on the topic *A survey of Lie group representations*. The present paper is partially an outgrowth of some of the ideas on multiplicities of representations stated during that talk; received by the editors March 3, 1969.