CATEGORIES OF V-SETS

BY J. A. GOGUEN[1]

Communicated by Saunders Mac Lane, December 9, 1968

Let V be a partially ordered set. Then a *V-set* is a function $A: X \to V$ from a set X to V. V is the set of *values* for A, and X is the *carrier* of A. If $B: Y \to V$ is another V-set, a *morphism* $f: A \to B$ is a function $\bar{f}: X \to Y$ such that $A(x) \leqq B(\bar{f}(x))$ for each $x \in X$. The category of all V-sets is denoted $\mathcal{S}(V)$. The *carrier functor* $K: \mathcal{S}(V) \to \mathcal{S}$ assigns X to $A: X \to V$ and $\bar{f}: X \to Y$ to $f: A \to B$, where $\mathcal{S}$ is the category of sets. See [2].

If V has one point, $\mathcal{S}(V) = \mathcal{S}$. If $V = \{0, 1\}$, where $0 < 1$, $\mathcal{S}(V)$ is the category of pairs (X, A) of sets, where $A \subseteq X$. If V is the closed unit interval, $\mathcal{S}(V)$ is the category of "fuzzy sets", as used by Zadeh and others [1], [5] for problems of pattern recognition and systems theory. When V is a Boolean algebra, V-sets are Boolean-valued sets, as used by Scott and Solovay for independence results in set theory (however, their notion of morphism is different).

If V is complete, $\mathcal{S}(V)$ is a pleasant category satisfying all Lawvere's axioms [3] for $\mathcal{S}$ except choice, modulo some substitutions of the V-set with carrier 1 and value 0 for the terminal object. In particular,

THEOREM 1. *If V is complete, $\mathcal{S}(V)$ is complete and cocomplete, has an exponential (i.e., a coadjoint to product) and a "Dedekind-Pierce object" (i.e., an object which looks like the set of integers; see [3]).*

Let Poc denote the category of partially ordered classes, and let $\mathcal{L}$ be a subcategory of Poc. Then a category $\mathcal{C}$ is *$\mathcal{L}$-ordered* if the *power function* $\mathcal{P}: |\mathcal{C}| \to$ Poc factors through $\mathcal{L}$, where $\mathcal{P}(A)$ is the class of all equivalence classes of monics with codomain A ($f \equiv g$ if $\exists$ an isomorphism h such that $fh = g$). Denote the image of $A \xrightarrow{f} B$ by $f(A)$, and the image of the composite $A' \xrightarrow{i} A \xrightarrow{f} B$, where i is monic, by $f(A')$. Then $\mathcal{C}$ has *associative images* if it has images such that $f(g(A)) = (fg)(A)$, whenever $A \xrightarrow{g} B \xrightarrow{f} C$. $\mathcal{P}$ can be construed as a functor when $\mathcal{C}$ has associative images. Let CL denote the category of complete lattices, and call a category C_1 if a coproduct of monics is always monic.

---

[1] Research supported by Office of Naval Research under contracts Nos. 3656(08) and 222(85), at the University of California at Berkeley.