ON DISCRETE BOREL SPACES AND PROJECTIVE SETS

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Let I denote the unit interval, $S=I\times I$ the unit square; C_I and C_S the class of all subsets of I and S, respectively. By $C_I\times C_I$ is meant the σ-algebra on S generated by rectangles with sides in C_I. The purpose of this note is to prove the following theorem (which settles a problem of S. M. Ulam) and observe some of its consequences. *Without explicit mention, the axiom of choice has been assumed throughout this paper.* CH stands for the continuum hypothesis.

THEOREM 1. *If CH is valid, then* $C_I\times C_I=C_S$.

PROOF. First, observe that if f is any function defined on a subset of I into I then its graph

$$G = \{(x, y): x \in \text{Domain of } f, f(x) = y\}$$

is in $C_I\times C_I$. For this it suffices to verify that

$$G = \bigcap_{n=1}^{\infty} S_n; \quad \text{where} \quad S_n = \bigcup_{k=1}^{n} \{A_{nk} \times B_{nk}\},$$

$$A_{nk} = \{x \in \text{Domain } f: (k-1)/n \leqq f(x) < k/n\},$$

$$B_{nk} = \{y \in \text{Range } f: (k-1)/n \leqq y < k/n\}.$$

(For $k=n$; include the right endpoint as well.)

Second, if $B\subset S$ be such that every vertical section is at most countable then $B\in C_I\times C_I$. This follows by realizing B as countable union of graphs.

Third, if $B\subset S$ is such that every horizontal section is at most countable then $B\in C_I\times C_I$.

Fourth, $S=X\cup Y$ where every vertical section of X is at most countable and every horizontal section of Y is at most countable [4]. This can be done by realizing I as the set of ordinals less than the first uncountable ordinal (by using CH) and then taking the portions below and not below the diagonal.

Finally, if $B\subset S$ then by previous remarks $B\cap X$, $B\cap Y$ are in $C_I\times C_I$ to complete the proof.

Let Z be a set of cardinality $\aleph_1$, the first uncountable cardinal. An