RATIONAL POINTS ON ALGEBRAIC VARIETIES OVER LARGE NUMBER FIELDS[1]

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Denote by Σ the class of all fields K which have the following property: For any nonvoid absolutely irreducible variety V defined by equations over K, the set of points of V rational over K is not empty.

For any prime p denote by F_p the field with p elements. Then it follows from the Riemann hypothesis for curves [1] that if $\mathfrak{F} = \prod F_p/D$ is a nonprincipal ultra-product of the F_p then $\mathfrak{F} \in \Sigma$ (see [2, Theorem 6]). On the other hand, it follows from the Hilbert Nullstellensatz that if K is an algebraically closed field then $K \in \Sigma$. In particular it follows that the algebraic closure of Q (the field of rational numbers), $\tilde{Q}$, belongs to Σ. It is therefore natural to ask whether or not $\mathfrak{F} \cap \tilde{Q} \in \Sigma$. Ax gave a counterexample in [3, §14], showing that this is not always the case. One can then ask whether Ax's example is the exception or the rule. We shall see, however, that Ax's example is exceptional and that in general, $\mathfrak{F} \cap \tilde{Q}$ does belong to Σ. To be more precise denote by $Q(\sigma)$ the fixed field in $\tilde{Q}$ of an automorphism $\sigma \in \mathcal{G}(\tilde{Q}/Q)$ ($\mathcal{G}(\tilde{Q}/Q)$ is the Galois group of $\tilde{Q}$ over Q). Ax showed ([2, Theorem 5]) that for every nonprincipal ultra-product $\mathfrak{F}$ of the F_p there exists $\sigma \in \mathcal{G}(\tilde{Q}/Q)$ such that $\mathfrak{F} \cap \tilde{Q} = Q(\sigma)$, and conversely, for each $\sigma \in \mathcal{G}(\tilde{Q}/Q)$ there exists a nonprincipal ultra-product $\mathfrak{F}$ of the F_p such that $\mathfrak{F} \cap \tilde{Q} \cong Q(\sigma)$. What we shall in fact prove is that for almost all $\sigma \in \mathcal{G}(\tilde{Q}/Q)$ (in the sense of Haar measure), $Q(\sigma) \in \Sigma$. More generally, let k be a field of characteristic zero. Denote by μ_k the normalized Haar measure defined on $\mathcal{G}(\tilde{k}/k)$ with respect to the Krull topology. For any positive integer s denote by μ_k^s the product measure defined on $\mathcal{G}(\tilde{k}/k)^s$. Then the following theorem is true.

THEOREM. *If k is a denumerable Hilbertian field of characteristic zero, then for almost all $(\sigma_1, \cdots, \sigma_s) \in \mathcal{G}(\tilde{k}/k)^s$ the fixed field of $\{\sigma_1, \cdots, \sigma_s\}$, $k(\sigma_1, \cdots, \sigma_s)$, belongs to Σ.*

Since it is well known that Q is a Hilbertian field (see e.g. [4]) we have in particular the following corollary.

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