A NOTE ON WEAKLY COMPLETE ALGEBRAS[1]

BY WILLIAM FULTON

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Fix a commutative noetherian ring R with unit and an ideal I in R. P. Monsky and G. Washnitzer have developed the notion of a weakly complete finitely generated algebra over (R, I) [1], [2]; we include a definition in §2. They have used these "w.c.f.g. algebras" to construct a p-adic De Rahm cohomology for nonsingular varieties defined over fields of characteristic p [1]. It is important for their theory that w.c.f.g. algebras are noetherian; we prove this fact here. Our proof attempts to follow the well-known proof that power series rings over R are noetherian. At one point we need a general lemma concerning modules over polynomial rings; §1 deals with this.

1. Let $R' = R[X_1, \cdots, X_n]$. The degree of a polynomial $f \in R'$ is denoted by ∂f. If S is a finitely generated free R'-module with a fixed basis, identify S with $(R')^m$, and for $f = (f_1, \cdots, f_m) \in S$, define $\partial f = \operatorname{Max} \partial f_i$.

LEMMA. *Let M be a submodule of S, S as above. Then M has a finite number of generators g_α so that any $g \in M$ may be written $g = \sum a_\alpha g_\alpha$ with $a_\alpha \in R'$ and $\partial a_\alpha \leqq \partial g - \partial g_\alpha$.*

PROOF. Let $R^* = R[X_0, X_1, \cdots, X_n]$, $S^* = (R^*)^m$. For each $f = (f_1, \cdots, f_m) \in S$, with $\partial f = d$, write $f^* = (f_1^*, \cdots, f_m^*) \in S^*$, where $f_i^* = X_0^d f_i(X_1/X_0, \cdots, X_n/X_0)$. Let M^* be the (homogeneous) submodule of S^* generated over R^* by $\{g^* \mid g \in M\}$. For the desired generators take any finite set of $g_\alpha \in M$ so that the g_α^* generate M^*. In fact, if $g \in M$, we may write $g^* = \sum A_\alpha g_\alpha^*$, and by homogeneity we may assume $A_\alpha \in R^*$ is of degree $= \partial g^* - \partial g_\alpha^* = \partial g - \partial g_\alpha$. Replacing X_0 by 1 in this equation shows that $g = \sum a_\alpha g_\alpha$, $a_\alpha = A_\alpha(1, X_1, \cdots, X_n)$, and $\partial a_\alpha \leqq \partial A_\alpha = \partial g - \partial g_\alpha$.

2. DEFINITION [2, §2.1]. An R-algebra A is a w.c.f.g. algebra over (R, I) if it satisfies the following two conditions:

(i) $\bigcap_{i=0}^{\infty} I^i A = 0$. We therefore identify A with its image under the natural map $A \to A^\infty = \text{proj lim}_i\, A/I^i A$.

(ii) There are elements $x_1, \cdots, x_n$ in A so that for any $y \in A$ there are polynomials $p_d(X_1, \cdots, X_n) \in I^d[X_1, \cdots, X_n]$ and a

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