SCHAUDER BASES IN SPACES OF DIFFERENTIABLE FUNCTIONS

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Banach [1, p. 238] states that Schauder bases are known for the spaces $C^k(I)$ but it is not known if $C^1(I\times I)$ has a Schauder basis. In this note we construct a Schauder basis for $C^1(I\times I)$.

1. **Definitions and notation.** We say that $\{x_n;\alpha_n\}$ (or simply $\{x_n\}$) is a Schauder basis for a Banach space X if for each $x\in X$ there exist unique scalars $a_i=\alpha_i(x)$ such that $x=\sum_{i=1}^{\infty} a_ix_i$ (i.e. the sequence of partial sums $\{\sum_{i=1}^{n} a_ix_i\}$ converges to x in norm).

It is well known [4] that each α_n is a continuous linear functional on X. Also, a total[1] set $\{x_n\}$ is a Schauder basis for X if and only if there exists a constant M such that

$$\left\|\sum_{i=1}^{p} a_ix_i\right\| \leqq M\left\|\sum_{i=1}^{p+q} a_ix_i\right\| \tag{1}$$

for any sequence $\{a_i\}$ of scalars and any natural numbers p, q. In the sequel we simply say "basis" for "Schauder basis".

We will denote by I the closed interval $[0, 1]$, by $C(I)$ the Banach space of real-valued continuous functions f defined on I with norm $\|f\|_\infty=\sup_{x\in I}|f(x)|$. $C^k(I)$ is the Banach space of real-valued f having k continuous derivatives with norm $\|f\|_k=\|f\|_\infty+\|f'\|_\infty+\cdots+\|f^{(k)}\|_\infty$. Finally, $C^1(I\times I)$ is the Banach space of real-valued functions $h=h(x, y)$ defined on $I\times I$ with continuous first partial derivatives. The norm for $C^1(I\times I)$ is given by

$$\|h\| = \sup_{(x,y)\in I\times I}|h(x,y)| + \sup_{(x,y)\in I\times I}\left|\frac{\partial}{\partial x}h(x,y)\right| + \sup_{(x,y)\in I\times I}\left|\frac{\partial}{\partial y}h(x,y)\right|.$$

2. **Construction of bases for $C^k(I)$.** Let $\{\phi_n;\mu_n\}$ be any basis for $C(I)$ and let

$$\begin{aligned} f_1(x) &= 1, & \alpha_1(f) &= f(0),\\ f_n(x) &= \int_0^x \phi_{n-1}(t)dt, & \alpha_n(f) &= \mu_{n-1}(f'), \end{aligned} \tag{2}$$

[1] total = finite linear combinations are dense in X.