GENERALIZATION OF THE JACKSON APPROXIMATION THEOREMS IN THE SENSE OF CH. MÜNTZ

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1. **Introduction.** The aim of this note is the generalization of the theorems of D. Jackson [1]–[5] for linear combinations $\sum_{i=0}^{s} a_i x^{p_i}$.

THEOREM 1 (CH. MÜNTZ [1], [2], [4]). *Let* $p_0, p_1, \cdots$ *be distinct real numbers such that* $0 \leq p_0 < p_1 < \cdots$ *and* $\lim_{i\to\infty} p_i = \infty$. *The set of powers* $\{x^{p_0}, x^{p_1}, \cdots\}$ *is fundamental (with the uniform norm) in* $C[0, 1]$ *if and only if* $p_0 = 0$ *and* $\sum_{i=1}^{\infty} 1/p_i = \infty$.

Considering this theorem we ask whether the error in the[2] best uniform approximation of f,

$$(1) \qquad \tilde{E}_s(f; \{p_i\}) := \min_{a_i} \left(\max_{x\in[0,1]} \left| f(x) - \sum_{i=0}^{s} a_i x^{p_i} \right| \right),$$

satisfies inequalities similar to those of the Jackson theorems for the error

$$(2) \qquad E_n(f) := \min_{a_i} \left(\max_{x\in[a,b]} \left| f(x) - \sum_{i=0}^{n} a_i x^i \right| \right)$$

when the exponents p_i are of the type of Theorem 1. Our problem is therefore to find a connection between the asymptotic behaviour of the error $\tilde{E}_s(f; \{p_i\})$ for $s \to \infty$, the sequence $\{p_i\}$, and the "smoothness" of the function f. We only present our main results here. The full details will be published elsewhere.

2. **Jackson theorems for polynomials** $\sum_{i=0}^{s} a_i x^{i\cdot r}$, $r > 0$. We consider the sequence $p_i = i\cdot r$, $i \in N \cup \{0\}$ and $r > 0$, where $N = \{1, 2, 3, \cdots\}$. Then we approximate the function $f \in C[0, 1]$ by polynomials $\tilde{P}_s(x) = \sum_{i=0}^{s} a_i x^{i\cdot r}$. We first assume that $f(x) = x^q$ and consider $\tilde{E}_s(x^q; \{i\cdot r\})$.

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[1] The results of this note were announced by the author in lectures held on September 16, 1967 at the German Mathematical Congress, Karlsruhe, on May 25, 1968 at the Bavarian Mathematical Congress, Eichstätt, and at the Mathematical Research Institute, Oberwolfach, Black Forest, in July 1968. The author is very grateful to Doz. Dr. P. O. Runck for helpful comments and suggestions.

[2] If $p_0 = 0$, then for each function $f \in C[0, 1]$ the polynomial $\sum_{i=0}^{s} a_i x^{p_i}$ of best approximation is unique since the set of functions $\{1, x^{p_1}, \cdots, x^{p_s}\}$ satisfies the Haar condition on $[0, 1]$.