LOCAL LEFT NOETHERIAN IPLI-RINGS

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All rings are associative and unitary. A ring R is a *pli-ring* (resp. ipli-ring) if every left ideal (resp. two-sided ideal) of R is of the form Ra for some $a \in R$. Clearly, every pli-ring is a left Noetherian ipli-ring. A ring R is called *local* if R has a unique maximal left ideal.

This note contains statements of some results concerning ideals and global dimensions of local left Noetherian ipli-rings.

A few definitions are needed. Let I be an ideal (i.e., two-sided ideal) of a ring R. We shall give two definitions of transfinite powers of I. The first is: $I^1 = I$; $I^\alpha = I \cdot I^\beta$ if $\alpha = \beta + 1$; $I^\alpha = \bigcap_{\beta<\alpha} I^\beta$ if α is a limit ordinal. The second definition is notationally distinguished from the first by writing the index ordinal in a square bracket; it goes as follows:

$$I^{[\omega^0]} = I; \qquad I^{[\omega^\alpha]} = \bigcap_{n=1}^{\infty} (I^{[\omega^\beta]})^n \quad \text{if } \alpha = \beta + 1; \qquad I^{[\omega^\alpha]} = \bigcap_{\beta<\alpha} I^{[\omega^\beta]}$$

if α is a limit ordinal. Note that the second definition defines transfinite powers only for ordinals of the form ω^α. For all the set-theory involved, we refer to [3].

The following theorem is basic.

THEOREM 1. *Let A be a proper prime ideal in a prime left Noetherian ipli-ring R. Then there exists an ordinal α such that $A^{[\omega^\alpha]} = (0)$. Let α be the first such ordinal. Then $A^{[\omega^\beta]} \subsetneqq A^{[\omega^\gamma]}$ if $\gamma < \beta \leqq \alpha$. The prime ideals of R contained in A are precisely those of the form $A^{[\omega^\beta]}$ where $\beta \leqq \alpha$.*

Recall that a *domain* is a (not necessarily commutative) ring without zero-divisors.

THEOREM 2. *Let R be a local semiprime left Noetherian ipli-ring with Jacobson radical J. Then*

(1) *R is a pli-domain.*

(2) *There exists an ordinal α such that $J^{[\omega^\alpha]} = (0)$. Let α be the first such ordinal. For every $\beta < \alpha$, choose $x_\beta \in R$ such that $J^{[\omega^\beta]} = Rx_\beta$.*

(3) *Every nonzero element r of R can be uniquely expressed as*

$$r = ux_{\beta_1}^{m_1} \cdots x_{\beta_s}^{m_s},$$

where s is a nonnegative integer, $m_i \in Z^+$, $\beta_1 < \cdots < \beta_s \leqq \alpha$ and u is a unit in R.