MAPPING CYLINDERS AND THE ANNULUS CONJECTURE

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Suppose f is an embedding of the n-sphere S^n into the $(n+1)$-sphere S^{n+1}; f is said to be *locally flat* at $x \in S^n$ if there is a neighborhood U of $f(x)$ in S^{n+1} such that the pair $(U, U \cap f(S^n))$ is homeomorphic to (E^{n+1}, E^n) where E^i is Euclidean i-space; i.e., there exists a homeomorphism $h: U \to E^{n+1}$ such that $h(U \cap f(S^n)) = E^n \equiv E^n \times 0 \subseteq E^n \times E^1 = E^{n+1}$. Brown [2], [3] has shown that if f is locally flat at each point of S^n, then the closure of each complementary domain of $f(S^n)$ in S^{n+1} is homeomorphic to an $(n+1)$-cell. One of the outstanding unsolved problems in topology of manifolds is the annulus conjecture. Suppose f, g are two locally flat embeddings (i.e., f and g are locally flat at each point of S^n) of S^n into S^{n+1} such that $f(S^n) \cap g(S^n) = \emptyset$. The connected submanifold A^{n+1} of S^{n+1} whose boundary is $f(S^n) \cup g(S^n)$ is called a *pseudo-annulus*. The annulus conjecture is that A^{n+1} is homeomorphic to $S^n \times [0, 1]$. If f, g are both either piecewise linear or differentiable maps or if $n \leqq 2$, then the conjecture is true.

This paper was motivated by an attempt to construct a counterexample to the annulus conjecture. Let $p: S^n \to S^n$ be a continuous map. The *mapping cylinder* of p, Map (p), is the decomposition space formed from the disjoint union $(S^n \times [0, 1]) \cup S^n$ by identifying $(x, 1)$ with $p(x)$ for each $x \in S^n$. The idea was to find a map $p: S^n \to S^n$ such that Map (p) is an $(n+1)$-manifold which is not homeomorphic to $S^n \times I$; for example, one might attempt to construct such a p by using a variation of Bing's example [1] of an upper semicontinuous decomposition of S^3 which yields S^3 but some of whose nondegenerate elements are spheres. By Proposition 2, Map (p) would be a pseudo-annulus and hence a counterexample. However, we show that this is impossible in dimension 3; i.e., if Map (p) is a manifold, then it is homeomorphic to $S^3 \times I$.

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Let $p: S^n \to S^n$ be a continuous map such that Map (p) is an $(n+1)$-manifold. Let $\pi: (S^n \times I) \cup S^n \to$ Map (p) be the natural projection.

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