# RELATIVE GROTHENDIECK RINGS[1]

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1. **Introduction.** Let $H$ be a subgroup of the finite group $G$, and let $\Omega$ be a field of characteristic $p$, where we assume $p \neq 0$ to avoid trivial cases. Form the free abelian group $\mathcal{A}$ on the symbols $[M]$, where $M$ ranges over the representatives of a full set of isomorphism classes of finitely generated left $\Omega G$-modules (hereafter called "$G$-modules" for brevity). Let $\mathcal{B}$ be the subgroup of $\mathcal{A}$ generated by all expressions $[M]-[L]-[N]$, where

$$0 \to L \to M \to N \to 0$$

ranges over all $H$-split exact sequences of $G$-modules. The *relative Grothendieck ring* $a(G, H)$ is defined as $\mathcal{A}/\mathcal{B}$, acquiring a ring structure by letting $[M][M'] = [M \otimes_\Omega M']$ where $G$ acts diagonally on the tensor product.

The structure of $a(G, H)$ has been investigated by the authors in two earlier articles [1], [2]. In the extreme case where $H=1$, the ring $a(G, 1)$ is just the ring of generalized Brauer characters of $G$. On the other hand, $a(G, G)$ is the representation ring of $G$, gotten by considering $G$-modules relative to direct sum. In general, $a(G, K) \cong a(G, H)$ if $H$ is a Sylow $p$-subgroup of $K$, and so there is no loss of generality in assuming hereafter that $H$ is a $p$-subgroup of $G$.

Let $k(G, H)$ be the ideal of $a(G, G)$ spanned by all $(G, H)$-projective $G$-modules. The *Cartan map*

$$\kappa: k(G, H) \to a(G, H)$$

is defined by $[M] \to [M]$, and as shown in [2], $\kappa$ is a monomorphism. Furthermore, the cokernel of $\kappa$ is a $p$-torsion abelian group when $H \Delta G$.

We have previously established

THEOREM 1 [1, THEOREMS 3.4 AND 4.4]. *If $H \Delta G$, where $H$ is a cyclic $p$-group, then $a(G, H)$ has a finite free $Z$-basis. Furthermore, if $G$ is a semidirect product $H \cdot A$, then there is a $Z$-isomorphism*

$$a(G, H) \cong a(H, H) \otimes_Z a(G, 1).$$

*This isomorphism is in fact a ring isomorphism when $G$ is the direct product $H \times A$.*

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