THE FIXED POINT INDEX AND ASYMPTOTIC FIXED POINT THEOREMS FOR k-SET-CONTRACTIONS

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1. **Introduction.** In 1955 G. Darbo [6] defined the measure of noncompactness, $\gamma(A)$, of a bounded subset A of a metric space (X, d): $\gamma(A) = \inf\{d>0 \mid A$ can be covered by a finite number of sets of diameter less than or equal to $d\}$. If (X, d) is a complete metric space, Darbo shows that for any decreasing sequence of closed, nonempty sets A_n with $\gamma(A_n)$ approaching 0, $\bigcap_{n\geq 1} A_n$ is compact and nonempty. If X is a Banach space, Darbo also demonstrates the crucial properties $\gamma(A+B) \leqq \gamma(A)+\gamma(B)$ and $\gamma(\text{convex closure } A) = \gamma(A)$.

If G is a subset of the metric space X_1 and f is a continuous map from G to a metric space X_2, Darbo calls f a k-set-contraction if $\gamma_2(f(A)) \leqq k\gamma_1(A)$ for A bounded and $A \subset G$. It is easy to show that k-set-contractions with $k<1$ are closed under composition and convex sums. Darbo proves that if G is a closed, bounded convex subset of a Banach space X and $f: G \to G$ is a k-set-contraction, $k<1$, then f has a fixed point.

An important example of a k-set-contraction, $k<1$, is a map of the form $U+C$, U a strict contraction (i.e. $\|Ux-Uy\| \leqq k\|x-y\|$, $k<1$) and C a compact map, both defined on a subset G of a Banach space X. F. E. Browder and the author [5] have recently defined (as a special case) a degree theory for mappings of the form $I-U-C$, so it is natural to ask if one can obtain a degree theory for mappings of the form $I-f$, f a k-set-contraction, $k<1$. In fact we will define a fixed point index for k-set-contractions on certain nice ANR's, and we will give direct generalizations of all properties of the classical fixed point index.

In another direction let X be a bounded, complete metric space and $f: X \to X$ a k-set-contraction, $k<1$. Using Darbo's results we can prove that $\text{cl}(\bigcap_{n\geq 1} f^n(X))$ is nonempty and compact. In general Browder [3] has suggestively called such maps asymptotically compact and has proved fixed point theorems about them. Such theorems have proved useful in studying ordinary differential equations. We generalize Browder's chief result to the context of k-set-contractions.

2. **The fixed point index for k-set-contractions.** Let us begin by recalling the basic properties of the classical fixed point index. Let X