DIFFERENTIABLE FUNCTIONS ON c_0

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If E and F are two Banach spaces, denote by $C^{p,q}(E, F)$, $0 \leqq q \leqq p \leqq \infty$, those functions in $C^p(E, F)$ whose derivatives of order less than or equal to q are bounded. Call a Banach space, E, $C^{p,q}$ smooth if there exists a nonzero $C^{p,q}$ function on E with bounded support. Then finite dimensional spaces are $C^{\infty,\infty}$ smooth and if an L_p space is C^q smooth it is also $C^{q,q}$ smooth. Although c_0 is known to possess a C^∞ (away from zero) norm as described in Bonic and Frampton [1], it is a consequence of the following theorem that c_0 is not $C^{2,2}$ smooth.

THEOREM. *Let $f \in C^1(c_0, R)$ with Df uniformly continuous. Then the support of f is unbounded.*

PROOF. If not then there would exist an $f \in C^1(c_0, R)$ such that $f(0)=1$, $f(x)=0$ for $\|x\| \geqq 1$ and Df is uniformly continuous. Pick N such that $\|h\| \leqq 1/N$ implies $\|Df(x+h)-Df(x)\| \leqq 1/2$. Then the mean value theorem gives that $|f(x+h)-f(x)-Df(x)(h)| \leqq 1/2\|h\|$ when $\|h\| \leqq 1/N$. Let A be the set of all x in c_0 such that 2^N-1 of the first 2^N components of x have absolute value $1/N$, the remaining component has absolute value less than or equal to $1/N$ and all the components after the first 2^N are zero. Since A is connected and even, we can pick inductively $h_1, \cdots h_N \in A$ such that $Df(h_1+ \cdots +h_{k-1}) \cdot (h_k)=0$ and $h_1+ \cdots +h_k$ has at least 2^{n-k} components equal to k/N. Then

$$\|h_1 + \cdots + h_N\| = 1$$

and

$$\begin{aligned} &|f(h_1 + \cdots + h_N) - f(0)| \\ &\qquad \leqq \sum_{k=1}^{N} |f(h_1 + \cdots + h_k) - f(h_1 + \cdots + h_{k-1}) \\ &\qquad\qquad - Df(h_1 + \cdots + h_{k-1})h_k| \leqq \sum_{k=1}^{N} \frac{1}{2}\|h_k\| = \frac{1}{2} \end{aligned}$$

which is a contradiction.

COROLLARY 1. *Let $f \in C^1(c_0, R)$ and Df be uniformly continuous. Then $f(\delta U)$ is dense in $f(U)$ for all bounded open sets U.*