RIESZ OPERATORS AND FREDHOLM PERTURBATIONS

BY MARTIN SCHECHTER

Communicated by B. Yood, June 21, 1968

1. **Introduction.** Let X be a Banach space, and let $B(X)$ denote the space of bounded linear operators on X. An operator $A \in B(X)$ is called a *Fredholm operator* if

1. $\alpha(A)$, the dimension of the null space $N(A)$ of A, is finite;
2. the range $R(A)$ of A is closed in X;
3. $\beta(A)$ the codimension of $R(A)$, is finite.

The set of Fredholm operators on X is denoted by $\Phi(X)$. An operator $E \in B(X)$ is called a *Riesz operator* if $E-\lambda \in \Phi(X)$ for all scalars $\lambda \neq 0$. For further discussion of such operators we refer to [1, p. 323], [2], [3], [4], [5], [9].

An operator $E \in B(X)$ is called a *Fredholm perturbation* if $A+E \in \Phi(X)$ for all $A \in \Phi(X)$. In this paper we investigate the connection between Riesz operators and Fredholm perturbations. Our work complements the results of [2], [3] and [6].

2. **Riesz operators.** Let $R(X)$ denote the set of Riesz operators on X.

LEMMA 1. *$E \in R(X)$ if and only if $I+\lambda E \in \Phi(X)$ for all scalars λ.*

PROOF. If $E \in R(X)$, the statement is true for $\lambda = 0$. Otherwise $E+I/\lambda \in \Phi(X)$. Hence $I+\lambda E \in \Phi(X)$. Conversely, if $\mu \neq 0$, then $\mu(I+E/\mu) \in \Phi(X)$ showing that $E+\mu \in \Phi(X)$.

The set $K(X)$ of compact operators on X is a closed, two-sided ideal in $B(X)$. Let π be the natural quotient map of $B(X)$ into $B(X)/K(X)$.

LEMMA 2 [7]. *$A \in \Phi(X)$ if and only if $\pi(A)$ is invertible in $B(X)/K(X)$.*

LEMMA 3 [9], [1]. *$E \in R(X) \Leftrightarrow \|\pi(E)^n\|^{1/n} \to 0$ as $n \to \infty$.*

For any two operators A, $B \in B(X)$ we shall write $A \cup_\pi B$ when $AB-BA$ is a compact operator on X. The reason for the notation is that $\pi(AB) = \pi(BA)$ in this case. Such operators are said to "almost commute."

LEMMA 4. *If $E \in R(X)$ and $K \in K(X)$, then $E+K \in R(X)$.*

PROOF. $\pi(E+K-\lambda) = \pi(E-\lambda)$.