ON THE CHARACTERISTIC ROOTS OF TOURNAMENT MATRICES

BY ALFRED BRAUER[1] AND IVEY C. GENTRY[1]

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A tournament matrix $A=(a_{ij})$ of order n is a matrix of zeros and ones whose main diagonal elements are zeros and all other elements satisfy $a_{ij}+a_{ji}=1$ for $i\neq j$. See, for instance [5].

Such matrices have recently been studied in a large number of papers. But not much seems to be known about their characteristic roots. Since they are nonnegative matrices whose two greatest row-sums are less than or equal to $n-1$ and $n-2$, respectively, it follows from [2] that they lie in the interior or on the boundary of the circle

$$|z| \leqq ((n-1)(n-2))^{1/2}.$$

In this paper, this result will be improved.

THEOREM. *Let A be a tournament matrix of order n with characteristic roots $\omega_1, \omega_2, \cdots, \omega_n$, and $R(\omega_\nu)$ the real part of ω_ν. Assume that*

$$|\omega_1| \geqq |\omega_2| \geqq \cdots \geqq |\omega_n|.$$

Then

$$-\tfrac{1}{2} \leqq R(\omega_\nu) \leqq \tfrac{1}{2}(n-1),$$

and more exactly

$$\omega_1 \leqq \tfrac{1}{2}(n-1) \quad \text{and} \quad |\omega_\nu| \leqq \left(\frac{n(n-1)}{2\nu}\right)^{1/2} \quad \text{for} \quad \nu \geqq 2.$$

PROOF. Let B be the symmetric matrix $\frac{1}{2}(A+A')$. All its main diagonal elements are zeros and all other elements equal $\frac{1}{2}$. Since B is a generalized stochastic matrix with row-sum $\frac{1}{2}(n-1)$, its greatest root is $\frac{1}{2}(n-1)$. Moreover, it follows from [3] that the nontrivial roots remain unchanged if we subtract from all the elements of each column the number $\frac{1}{2}$. We obtain the diagonal matrix $D(-\frac{1}{2}, -\frac{1}{2}, \cdots, -\frac{1}{2})$. Hence B has the root $\frac{1}{2}(n-1)$ and $n-1$ roots $-\frac{1}{2}$.

In 1902, I. Bendixson [1] proved the following theorem.

Let T be a matrix with real elements, S the symmetric matrix $\frac{1}{2}(T+T')$, and M and m the maximum and the minimum of the char-

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