EXTENDING AUTOMORPHISMS ON PRIMARY GROUPS

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Let G be a p-primary abelian group, that is, let G be a commutative group each element of which has finite order equal to a power of a fixed prime p. We suppose that G is written additively and we say that an element $g \in G$ is divisible by a positive integer n if there is a solution in G to the equation $nx = g$. Note that 0 is divisible by every positive integer and that the collection of all the elements of G with this property is a subgroup of G. This subgroup is denoted by $p^\omega G$. It is immediate that $p^\omega(G/p^\omega G) = 0$. We can define $p^\alpha G$ for any ordinal α inductively by taking intersections at limit ordinals and letting $p^{\alpha+1}G = p(p^\alpha G)$.

It has been known for a long time that if the primary group G is countable, then any automorphism of $p^\omega G$ can be extended to an automorphism of G; this result is implicitly contained in Zippin's proof [7] of Ulm's theorem. It is worth noting that if G is countable, then $G/p^\omega G$ is a direct sum of cyclic groups according to a classical result due to Prüfer [6]: if G is a countable primary group such that $p^\omega G = 0$, then G must be a direct sum of cyclic groups. This suggests the possibility of extending Zippin's result to the case where G is any primary group with the property that $G/p^\omega G$ is a direct sum of cyclic groups. Recently, this result has been obtained by Hill and Megibben [3], [4], in a more general setting, and independently by Crawley [1]. What seems to the author as being a more striking generalization is the result being announced in this note, which says that the subgroup does not have to be $p^\omega G$ in order to extend height-preserving automorphisms.

By the height, $h_G(x)$, of an element x in G we mean the largest ordinal α such that $x \in p^\alpha G$ if such a largest ordinal α exists. Since $p^\beta G = \bigcap_{\alpha<\beta} p^\alpha G$ when β is a limit, the only failure of the existence of such a largest ordinal is when $x \in p^\alpha G$ for all α. In that case, we put $h_G(x) = \infty$ and adopt the convention that $\infty > \alpha$ for all ordinals α. If π is an automorphism of a subgroup H of G, we say that π is height-preserving in G if $h_G(x) = h_G(\pi(x))$ for all x in H. Denote by $\mathfrak{A}^G(H)$ the group of automorphisms of H that preserve heights in G and let $\mathfrak{A}_H(G)$ denote the group of automorphisms of G that map H onto H. Naturally, we write $\mathfrak{A}(G)$ for $\mathfrak{A}^G(G)$ since the latter is the full automorphism group of G, and we write $\mathfrak{A}(G)$ instead of $\mathfrak{A}_0(G)$ for the