A HOMOLOGICAL METHOD FOR COMPUTING CERTAIN WHITEHEAD PRODUCTS

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1. Introduction. In its simplest form the method for calculating the Whitehead product (WP) $\pi_{n_1}(X)\otimes\pi_{n_2}(X)\to\pi_{n_1+n_2-1}(X)$ may be described as follows. Suppose X is embedded in an H-space E so that the pair (E, X) has trivial homotopy groups in dimensions $<n_1+n_2$. Then we prove that the WP $[\alpha_1, \alpha_2]$ of $\alpha_1\in\pi_{n_1}(X)\equiv\pi_{n_1}(E)$ and $\alpha_2\in\pi_{n_2}(X)\equiv\pi_{n_2}(E)$ is the image under a homomorphism $H_{n_1+n_2}(E)\to\pi_{n_1+n_2-1}(X)$ of the Pontrjagin product of $h(\alpha_1)$ and $h(\alpha_2)$ in the homology ring $H_*(E)$, where $h\colon \pi_*(E)\to H_*(E)$ denotes the Hurewicz homomorphism. Thus, to determine $[\alpha_1, \alpha_2]$, it is necessary to know (1) the effect of h on α_1 and α_2, (2) the Pontrjagin product of $h(\alpha_1)$ and $h(\alpha_2)$, (3) the homomorphism $H_{n_1+n_2}(E)\to\pi_{n_1+n_2-1}(X)$.

It is, however, only sometimes possible to find an H-space for which the information (1), (2) and (3) is available. As a first example, consider the classifying space BU_t of the unitary group U_t and the WP

$$\pi_{2r+2}(BU_t)\otimes\pi_{2s+2}(BU_t)\to\pi_{2t+1}(BU_t),\ t=r+s+1.$$

Here we embed BU_t in the H-space BU_∞ and note that the required information is known. In this way we obtain a new proof of a theorem of Bott [1]. For a second example suppose $\pi_i(X)=0$ for $i<n$ and $n<i<2n-1$ and $\pi_n(X)=\pi$, where n is odd. Then X can be embedded in $K(\pi, n)$. The Pontrjagin square in $H_{2n}(\pi, n)$ is zero and so $[\alpha, \alpha]=0$ for any $\alpha\in\pi$. This result is due to Meyer and Stein [8] (see also §3).

We actually generalize the preceding method by considering kth order WP's instead of ordinary WP's and by requiring that there exist a pair (E, A) with A operating on E rather than an H-space E. Our main result Theorem 1 then yields for ordinary WP's ($k=2$) both the assertion of the first paragraph and a theorem of Meyer [4]. For $k>2$ it enables us, in §3, to extend Bott's theorem by computing kth order WP's in $\pi_*(BU_t)$, and to examine in some detail the kth order WP

$$\pi_n(X)\otimes\cdots\otimes\pi_n(X)\to\pi_{kn-1}(X)$$

when $\pi_i(X)=0$ for $i<n$ and $n<i<kn-1$.

Details of these results and other applications will appear elsewhere.