OPEN PROBLEMS ON UNIVALENT AND MULTIVALENT FUNCTIONS[1]

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1. **Introduction.** Let $f(z)$ be regular in the unit circle $\mathcal{E}$: $|z|<1$, and represented by the power series

$$w = f(z) = \sum_{n=0}^{\infty} b_n z^n = b_0 + b_1 z + b_2 z^2 + \cdots . \tag{1.1}$$

The function f maps $\mathcal{E}$ onto some subdomain $\mathcal{S}$ of a Riemann surface, and $\mathcal{S}$ is determined by the sequence $\{b_n\}$ of coefficients in (1.1).

Many questions (both open and settled) can be classified as special cases of the two general questions:

Given a geometric property of $\mathcal{S}$ what can be said about the sequence $\{b_n\}$? Given some information about the sequence $\{b_n\}$, what can be said about the domain $\mathcal{S}$?

One geometric property of $\mathcal{S}$ is specified by saying that $f(z)$ is *univalent* in $\mathcal{E}$. By definition, $f(z)$ is univalent in $\mathcal{E}$ if

$$f(z_1) = f(z_2), \qquad z_1, z_2 \in \mathcal{E} \Rightarrow z_1 = z_2. \tag{1.2}$$

Briefly, $f(z)$ is univalent in $\mathcal{E}$ if it assumes no value more than once for z in $\mathcal{E}$. When $f(z)$ is univalent, the image of $\mathcal{E}$ forms a *simple* domain in the w-plane. The concept of univalence has a natural extension as described in

DEFINITION 1. *Let p be a natural number. The function $f(z)$ is said to be p-valent (or multivalent of order p) in $\mathcal{E}$ if the conditions*

$$f(z_1) = f(z_2) = \cdots = f(z_{p+1}), \qquad z_1, z_2, \cdots, z_{p+1} \in \mathcal{E} \tag{1.3}$$

imply that $z_j = z_k$ for some pair such that $j \neq k$, and if there is some w_0, such that the equation $f(z) = w_0$ has p roots (counted in accordance with their multiplicities) in $\mathcal{E}$.

In brief, $f(z)$ is p-valent in $\mathcal{E}$ if it assumes no value more than p times in $\mathcal{E}$, but assumes some value p times in $\mathcal{E}$. We let $\mathcal{V}(p)$ denote the class of all functions that are regular and p-valent in $\mathcal{E}$, and have $f(0)=0$.

Certain related classes are also of interest. We let $\mathcal{K}(p)$ denote the subclass of $\mathcal{V}(p)$ of those functions f for which $f(\mathcal{E})$ is (in a generalized

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