FREDHOLM MAPS AND $GL_c(E)$-STRUCTURES

BY K. D. ELWORTHY

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Many of the difficulties in the study of functions between infinite dimensional Banach spaces disappear when one considers only perturbations of a fixed, well behaved, map by a class of maps with some finiteness condition on their range, for example compact perturbations of the identity map as in the Leray-Schauder theory. The results stated below are intended to indicate how this procedure can be extended to study maps between Banach manifolds. In particular it can be used to describe the homotopy properties of Fredholm maps, introduced by Smale in [5].

These results are contained in the author's Oxford doctoral thesis, written under the supervision of M. F. Atiyah. It is a great pleasure to be able to thank Professor Atiyah, and also Professor J. Eells for all their help and encouragement.

A version of Theorem 2 was proved independently by A. J. Tromba who used it to develop an oriented degree theory for proper Fredholm maps which he applied to give a proof of the Schauder existence theorem for quasi-linear elliptic equations. A detailed discussion of all these results is intended in a future joint publication with A. J. Tromba.

Throughout, E and F will denote infinite dimensional Banach spaces, and X a paracompact space. A C^p-smooth manifold will mean a C^p Banach manifold which admits C^p partitions of unity. For background material and an exhaustive bibliography see the survey article by Eells [3].

1. Linear theory. The nonlinear theory is based on the linear theory sketched here.

$L(E, F)$ will denote the Banach space of bounded linear maps $T: E \to F$, $\Phi_n(E, F)$ the subspace of Φ_n-operators (i.e. Fredholm operators of index n), $GL(E)$ the group of units in $L(E, E)$, $GL_c(E)$ the subgroup of $GL(E)$ consisting of elements of the form $I+\alpha$, where α is compact, and $GL_F(E)$ the corresponding group with α of finite rank. A vector bundle map which is a Φ_n-operator on each fibre will be called a 'Φ_n bundle map'.

PROPOSITION 1. *Let $\pi: B \to X$ be an E-vector bundle.*

(i) *A Φ_0 bundle map $f: B \to X \times E$ over the identity map of X induces a unique $GL_F(E)$-structure $\{\pi, f\}$ on π such that, in a trivialization of*