# ANALOGUES OF ARTIN'S CONJECTURE[1]

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Artin's celebrated conjecture on primitive roots (Artin [1, p. viii], Hasse [2], Hooley [3]) suggests the following

*Conjecture.* Let $S'$ be a set of rational primes. For each $q \in S$, let $L_q$ be an algebraic number field of degree $n(q)$. For every square-free integer $k$, divisible only by primes of $S$, define $L_k$ to be the composite of all $L_q$, $q \mid k$, and denote $n(k) = \deg(L_k/Q)$. Assume that $\sum_k 1/n(k)$ converges, where the sum is over those $k$ for which $L_k$ is defined. Then the natural density of the set $P$ of all primes $p$ which do not split completely in each $L_q$ exists and has the value $\sum_k \mu(k)/n(k)$, where $\mu$ is the Möbius function and the term $k=1$ has been included with $n(1)=1$.

If $S = \{\text{all rational primes}\}$, $L_q = Q(\zeta_q, a^{1/q})$, $a \in Z$, $\zeta_q =$ a primitive $q$th root of 1, then the conjecture is equivalent to Artin's conjecture. If $S$ is a finite set, then the conjecture is easily verifiable using the prime ideal theorem. For $S = \{\text{all rational primes}\}$, $L_q = Q(\zeta_{q^r})$, the conjecture has been proved by Knobloch [4] (for $r=2$ and only for Dirichlet densities) and by Mirsky [5].

We have proved the following theorems, whose proofs will appear elsewhere.

THEOREM 1. *Let there exist a finite set $S_0 \subset S$ such that $L_q \supset Q(\zeta_{q^2})$ for $q \in S - S_0$, and $L_q/Q$ is normal for all $q \in S$. Then the conjecture is true.*

THEOREM 2. *Suppose that for each finite subset $S_0 \subset S$ there exists a family of algebraic number fields $\{L_q'\}_{q \in S}$ such that*

(1) *$L_q = L_q'$ for $q \in S_0$,*

(2) *$L_q' \subset L_q$ for all $q \in S$,*

(3) *$L_q' \neq Q$ for all $q \in S$,*

(4) *the conjecture is true for $\{L_q'\}_{q \in S}$.*

*Then the conjecture is true for $\{L_q\}_{q \in S}$.*

THEOREM 3. *If the density $d(P)$ of $P$ exists, then*

$$d(P) \leq \sum_k \mu(k)/n(k).$$

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