BOUNDEDNESS OF SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS

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Communicated by L. Cesari, January 2, 1968

In the case of a linear constant coefficient differential equation, $\dot{x}=Ax$, where x is a (complex) n-vector and A is a (complex) $n\times n$ matrix, it is well known when all solutions are bounded; namely, if all eigenvalues of A are purely imaginary and all elementary divisions of A are simple. This condition is equivalent to the Jordan normal form, J, of A being (Hermitian) skew symmetric. That is if $J=PAP^{-1}$, then

$$J + J^* = PAP^{-1} + P^{*-1}A^*P^* = 0, \tag{1}$$

where M^* denotes the adjoint or complex conjugated transpose of M. Multiplying (1) on the left by P^* and on the right by P yields the equivalent condition that there exist a positive definite $Q=P^*P$ such that

$$QA + A^*Q = 0. \tag{2}$$

In the time dependent case, it is shown here that a necessary and sufficient condition that all solutions of $\dot{x}=A(t)x$ be bounded is that there exist a $Q(t)$ that is uniformly bounded and uniformly positive definite and that satisfies

$$Q(t)A(t) + A^*(t)Q(t) + \dot{Q}(t) = 0.$$

We will use the following notation. If ξ and η are complex n-vectors, then

$$\langle \xi, \eta\rangle = \sum_{i=1}^{n} \xi_i\bar{\eta}_i$$

will denote the inner product of ξ and η, and

$$\|\xi\| = \langle \xi, \xi\rangle^{1/2}$$

will denote the norm of ξ. Also M^* will denote the adjoint matrix of a matrix M.

The author would like to thank James J. Hurt for several helpful discussions during this work.

THEOREM 1. *Let $A(t)$ be an $n\times n$ matrix function defined and continuous on an open interval I. If there exists a continuously differentiable*