ON THE PROJECTION OF A SELFADJOINT OPERATOR

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In the present note some new properties of selfadjoint operators are given. These results came to light in our investigation of Weinstein's new maximum-minimum theory of eigenvalues [1], [2], but may be of some general interest by themselves, since they do not refer explicitly to eigenvalues. For the theory of unbounded operators, as used here, see the recent book of Goldberg [3].

Let T be any selfadjoint linear operator on a dense subspace $\mathfrak{D}$ of a Hilbert space $\mathcal{H}$. Let $\mathcal{Y}$ be any closed subspace of $\mathcal{H}$, let Y be the projection operator onto $\mathcal{Y}$, and let $\mathfrak{X}=\mathcal{Y}^{\perp}$ be the orthogonal complement of $\mathcal{Y}$ in $\mathcal{H}$. Then we have the following results.

LEMMA 1. *If $\mathfrak{X}$ is finite-dimensional, then the operator $T_0=YTY$ is selfadjoint.*

PROOF. Let $\mathfrak{D}_0$ denote the domain of T_0. Clearly $\mathfrak{D}_0=\mathfrak{X}\oplus(\mathfrak{D}\cap\mathcal{Y})$. The fact that $\mathfrak{D}_0$ is dense in $\mathcal{H}$ follows from a lemma of Gohberg and Krein [4], see also Goldberg [3, p. 103], which states that the intersection of a dense subspace and a closed subspace having finite deficiency is dense in the latter. Since T_0 is symmetric, it suffices to show that $\mathfrak{D}_0=\mathfrak{D}_0^*$, where $\mathfrak{D}_0^*$ denotes the domain of the adjoint of T_0. First of all, $u\in\mathfrak{D}_0$ implies that $Yu\in\mathfrak{D}=\mathfrak{D}^*$. Therefore (Tv, Yu) is continuous for all $v\in\mathfrak{D}$. In particular, (TYv, Yu) is continuous for all $Yv\in\mathfrak{D}$, so that $(YTYv, u)$ is continuous for all $v\in\mathfrak{D}_0$. This means that $u\in\mathfrak{D}_0^*$, and therefore $\mathfrak{D}_0\subset\mathfrak{D}_0^*$. On the other hand, let $u\in\mathfrak{D}_0^*$. Then $(YTYv, u)=(TYv, Yu)$ is continuous for all $v\in\mathfrak{D}_0$ or equivalently, (Tw, Yu) is continuous for all $w\in\mathfrak{D}\cap\mathcal{Y}$. Now let $\mathfrak{X}_1=\{x\in\mathfrak{X}\mid \exists y\in\mathcal{Y}$ for which $x+y\in\mathfrak{D}\}$. $\mathfrak{X}_1$ is clearly a subspace (not necessarily proper) of $\mathfrak{X}$ and therefore finite-dimensional.[2] Let $x_1, x_2, \cdots, x_s$ be any orthonormal basis for $\mathfrak{X}_1$ and let $y_1, y_2, \cdots, y_s$ be any elements in $\mathcal{Y}$ such that $z_j=x_j+y_j\in\mathfrak{D}$, $j=1, 2, \cdots, s$. Let $\mathcal{Z}=\mathrm{sp}\{z_1, z_2, \cdots, z_s\}$. It is easy to see that $\mathfrak{D}=\mathcal{Z}+(\mathfrak{D}\cap\mathcal{Y})$. Now let $\mathcal{W}_0=\mathfrak{D}\cap\mathcal{Y}$ and

$$\mathcal{W}_j = \mathrm{sp}\{z_1, z_2, \cdots, z_j\} + \mathcal{W}_0, \qquad j = 1, 2, \cdots, s.$$

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[2] It can be shown that $\mathfrak{X}_1=\mathfrak{X}$ but this fact is not needed for the proof of the lemma.