THE NONORIENTABLE GENUS OF K_n

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1. Introduction. One of the oldest problems in combinatorics is that of determining the chromatic number of each nonorientable 2-manifold. The problem is equivalent to determining the nonorientable genus of each complete graph, and was solved by Ringel [1] during the last decade using rather complicated methods.

A determination of the nonorientable genus of K_n by quite simple combinatorial techniques was presented in [2] if $n \equiv 3$, 4 or 5 (mod 6). This note is concerned with the situation in case $n \equiv 0$, 1 or 2 (mod 6).

The solution in these cases is not as elegant as that obtained for the other residue classes modulo 6. There certain combinatorial properties of the cyclic group Z_{6t+3} led to an extraordinarily unified solution. One might hope that using the group Z_{6t} would produce similar unification here. This is not the case because the two groups have significantly different combinatorial properties. In fact the solution presented here divides each case into two subcases. To be more precise, the nonorientable genus of K_n is determined for $n = 12s + k$, where $k = 0$, 6; 1, 7; 2, 8.

Details are offered for $k = 0$, 7 and 8; the first two because of an interesting comparison which can be made with the companion problem of determining the orientable genus of K_n. In the orientable problem (not yet solved for all n) the case $k = 0$ was particularly difficult, here it is almost trivial; in the orientable problem the case $k = 7$ is very simple, here it is somewhat involved. The case $k = 8$ is presented because Ringel found it particularly troublesome.

2. Definitions and general comments. In order to save space the reader is referred to [2] for the basic definitions and ideas involved.

Of particular importance is $\tilde{\gamma}(K_n)$, the nonorientable genus of the complete n-graph K_n, and

$$\tilde{I}(n) = \{(n-3)(n-4)/6\},^1 \qquad n = 5, 6, 7, \cdots.$$

We show that if $n \neq 7$ then

$$\tilde{\gamma}(K_n) = \tilde{I}(n). \tag{1}$$

[1] $\{a\}$ is the smallest integer not less than a.