NOTE ON CLASSES OF FUNCTIONS DEFINED BY INTEGRATED LIPSCHITZ CONDITIONS[1]

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The classes of functions with period 2π defined in terms of specific integrated Lipschitz conditions were characterized also in terms of degree of mean trigonometric approximation by Hardy and Littlewood without proof, results proved later by Quade [1], and still later supplemented by Zygmund [2] in his study of smooth functions. Analogs on approximation by complex polynomials of these results are due to Walsh and Russell [3]. The relative inclusion properties of these classes are difficult to treat directly, but certain results can be readily obtained by means of polynomial approximation properties, as is the purpose of this note to indicate.

A function $f(w)$ analytic in $|w|<1$ is said (Hardy) to be *of class* H_p $(p>1)$ there if the pth power norms of $f(re^{i\phi})$ with r fixed are uniformly bounded for $0<r<1$; under these conditions, boundary values $f(e^{i\phi})$ for approach "in angle" as $r\to 1$ exist for almost all ϕ, and $\int_0^{2\pi}|f(e^{i\phi})|^p d\phi$ exists. A function $f(z)$ analytic in the interior of an analytic Jordan curve C in the z-plane is *of class* H_p $(p>1)$ there if its transform is of class H_p when the interior of C is mapped onto $|w|<1$. Such a function is *of class* $H_p(k, \alpha)$ on C, $p>1$, $0<\alpha<1$, provided $f^{(k)}(z)$ exists on C and

$$\int_C |f_1^{(k)}(s+h) - f_1^{(k)}(s)|^p |dz| \leqq A|h|^{p\alpha}, \tag{1}$$

where s indicates arc length on C and $f_1^{(k)}(s)$ is the kth derivative of $f(z)$ with respect to s on C. Here and below A represents a constant independent of s and z which may change from one inequality to another. The Zygmund condition for $\alpha=1$ replaces (1) by

$$\int_C |f_1^{(k)}(s+h) + f_1^{(k)}(s-h) - 2f_1^{(k)}(s)|^p |dz| \leqq A|h|^p, \quad p>1, \tag{2}$$

and defines functions (of class H_p) *of class* $H_p(k, 1)$.

An analog [3] of the results on trigonometric approximation already mentioned is

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