REPRESENTATION OF NONNEGATIVE CONTINUOUS FUNCTIONS ON PRODUCT SPACES

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The purpose of this note is to show that a nonnegative continuous function on a locally compact σ-compact product space is a countable sum of products of factor functions. That is, if f is a nonnegative continuous function on $\times_{\alpha\in A} X_\alpha$, there exist nonnegative continuous functions $g_{\alpha i}$, such that for each i, only finitely many $g_{\alpha i}$ are not identically 1, and for all x,

$$f(x) = \sum_i \prod_\alpha g_{\alpha i}(x_\alpha).$$

Since we are dealing with continuous real-valued functions, we may assume the spaces are completely regular. By local compactness and σ-compactness, we can reduce the problem to a countable number of representations on products of compact spaces. Each of these spaces can be represented as a closed subspace of a product of cubes. Then the function f^* defined on the product of the embeddings can be extended to the product of the cubes. Consequently, it is sufficient to prove the theorem when all X_α are $[0, 1]$.

Define the function

$$h(x) = 1 - |x|, \qquad |x| < 1,$$
$$= 0, \qquad |x| \geqq 1. \tag{1}$$

Then if $0 \leqq x \leqq 1$,

$$1 = \sum_{i=0}^{n} h(nx - i). \tag{2}$$

Let us proceed inductively as follows. Let $f_0 = f$, and when f_i is defined, $M_i = \max_x f_i(x)$. Then there exist $\alpha_1, \cdots, \alpha_q$, n such that

$$|f_i(x) - f_i(y)| < M_i/4 \text{ whenever } |x_{\alpha_t} - y_{\alpha_t}| < 1/n,\ t = 1, \cdots, q.$$

Now let

$$k_i(x) = \sum_{j_1=0}^{n} \cdots \sum_{j_q=0}^{n} \prod_{t=1}^{q} h(nx_{\alpha_t} - j_t) \max(0, f_i(u_j) - \tfrac{1}{4}M_i), \tag{3}$$

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