EXTENDING LIAPUNOV'S SECOND METHOD TO NON-LIPSCHITZ LIAPUNOV FUNCTIONS

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The behavior of solutions of an ordinary differential equation

$$\text{(E)} \qquad dx/dt = f(t, x),$$

where $f: U \to R^n$ is continuous on the open set $U \subset R \times R^n$, is frequently studied by means of a continuous function $V: U \to R$. It is sometimes unnecessary to know the solutions explicitly. If for example V is independent of t, $V(x_0) = 0$ for some x_0, $V(x) > 0$ for $x \neq x_0$, and if for each solution ϕ of (E), $V(\phi(t))$ is a monotonically decreasing function of t for $t \geqq 0$, then x_0 is a stable critical point of (E). For V a C^1 function, Liapunov defined

$$\dot{V}(t, x) = \frac{\partial}{\partial t} V(t, x) + \langle \operatorname{grad}_x V(t, x), f(t, x) \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in R^n. He observed that for any solution ϕ,

$$\dot{V}(t, \phi(t)) = dV(t, \phi(t))/dt;$$

hence the rate of change of $V(t, \phi(t))$ can be calculated directly from V and f without knowing the solutions when V is a C^1 function. Sometimes a likely function V is not C^1, and for converse theorems frequently the most difficult problem is proving V can be chosen to be a smooth function. A theory was thus developed for $V \in C^0$ (V locally Lipschitz in x) primarily by Yoshizawa [2, p. 4] with earlier results by Okamura [1]. We will mean by $\phi(\cdot\,; t_0, x_0)$ that ϕ is a solution of (E) such that $\phi(t_0) = x_0$. When we refer to the domain of ϕ, we will assume that ϕ cannot be extended to a larger domain and still be a solution. Define for a solution $\phi = \phi(\cdot\,; t, x)$

$$\text{(1)} \qquad V'(t, x) = \liminf_{\tau \to +0} \tau^{-1}[V(t + \tau, \phi(t + \tau)) - V(t, \phi(t))],$$

$$\text{(2)} \qquad \dot{V}(t, x) = \liminf_{\tau \to +0} \tau^{-1}[V(t + \tau, x + \tau f(t, x)) - V(t, x)].$$

In (1) and (2) we use the so-called lower right-hand Dini derivate. For $V \in C^0$, if W is a continuous real-valued function and $\dot{V}(t, x)$

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