ON SOME FIXED POINTS THEOREMS IN GENERALIZED COMPLETE METRIC SPACES

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In Theorem 2 of [1][2] A. F. Monna generalized a result by W. A. J. Luxemburg on fixed points [2], valid for one operator in a generalized complete metric space, to a suitable family of operators; this result was later completed by M. Edelstein [3].

Clearly, when the family reduces to a unique element (i.e. $T_i \equiv T$ for all i), one gets Luxemburg's result. But if one considers the family of iterates of T: $T_i = T^i$ ($i = 1, 2, \cdots$), since Hypothesis 1 of Monna's Theorem requires $d(T_i x, T_i y) \leqq \rho d(x, y)$ ($i = 1, 2, \cdots$) when $d(x, y) \leqq C$, we must have, in particular, $d(Tx, Ty) \leqq \rho d(x, y)$, and Luxemburg's Theorem applies, providing even with a stronger conclusion than Monna's for this particular situation. In order to include this case as a strict generalization of Luxemburg's result, we relax Hypothesis 1 slightly, thus including also Monna's Theorem, and at the same time we get for the family $\{T^i\}$ a nontrivial result. This last assertion will be clarified with an example. This constitutes §1 of our paper.

In §2 we give some fixed point results for a family of operators with $\rho = 1$.

1. THEOREM 1. *Let (X, d) be a generalized complete metric space,*[3] *and $\{T_i\}_{i=1,2,\cdots}$ a family of self-mappings of X, closed under composition, such that*

(1) *There exist constants $C > 0$, $0 \leqq \rho < 1$, and an integer $m \geqq 1$ such that if $x, y \in X$ and $d(x, y) \leqq C$, then*

$$d(T_{m+k}x, T_{m+k}y) \leqq \rho d(x, y); \qquad k = 0, 1, 2, \cdots.$$

(2) $T_i = T_j = T_j T_i$; $i, j = 1, 2, \cdots$.

(3) *Let $x_0 \in X$ be arbitrary, and define $x_n = T_n x_{n-1}$ ($n = 1, 2, \cdots$). Then there exists $N(x_0)$ such that $d(T_{n+k}x_n, x_n) \leqq C$, for $n \geqq N$, $k = 1, 2 \cdots$.*

Then, there exists a $\xi \in X$ such that $x_n \to \xi$ and $T_n \xi \to \xi$ as $n \to \infty$.

Furthermore, (if)

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[2] Numbers in [ ] correspond to References.

[3] We follow Luxemburg's denomination.