THE FOLDED RIBBON THEOREM FOR REGULAR CLOSED CURVES IN THE PLANE

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Let S be the oriented circle with base point, E the oriented Euclidian plane, and V the positively oriented two frames in E. Let L be the space of C^1-regular immersions $g: S \to E$ with continuous right transverse field $\hat{g}$. For $g \in L$, set $\partial g = (g, \hat{g}, g'): S \to E \times V$. A positive monotone regular homotopy (=monotopy) from loop g_{-1} to g_{+1} is a C^1-regular homotopy G: $[-1, +1] \times S \to E$ with positive Jacobian and $\partial G(i, x) = \partial g_i(x)$, $i = \pm 1$, where $\partial G = (G, \partial G/\partial t, \partial G/\partial x)$. A negative monotopy G from g_{-1} to g_{+1} is such that $G^*(t, x) = G(-t, x)$ is a positive monotopy from g_{+1} to g_{-1}. A monotopy is stronger than a regular homotopy in that the latter requires only that $\partial G/\partial x \neq 0$. The tangent winding number TWN of g in L is the degree of $g'/|g'|: S \to S$. Because degree is a homotopy invariant, regular homotopy preserves the TWN. The converse of this is the Whitney-Graustein Theorem [3]. The TWN actually classifies L in a much stronger fashion.

THEOREM. *For two regular loops g_i, $i = \pm 1$, of like TWN, there always is a regular loop g_0 and two monotopies H_i: $g_i \sim g_0$, $i = \pm 1$, of like sign equal to sign $(TWN \pm \frac{1}{2})$.*

Note that TWN $= 0$ belongs to both cases. For TWN $= 1$, two concentric circles are monotopic. Not so for two circles whose interiors are disjoint; yet each is monotopic to a circle surrounding them both.

The method of proof is entirely constructive. The normal loops L_N have only simple, signed, transverse self-intersections (=nodes). L_N is dense and open (=generic) in L under the topology induced by $\|g-h\| = \max |\partial g(x) - \partial h(x)|$, $x \in S$. (See [3] for details.)

PROPOSITION 1. *If $g \in L$ and $\epsilon > 0$, there is an $h \in L_N$ with $\|g-h\| < \epsilon$ and a monotopy of prescribed sign between them.*

The proof of Proposition 1 makes use of a stable condition of "parallelity": min det $(g(x) - h(x),\ tg'(x) + (1-t)h'(x)) > 0$, over all $x \in S$ and $t \in [0, 1]$. The key lemma reads:

LEMMA. *If w is a continuous, periodic, transverse field along the ordinate in the (t, x)-Cartesian plane, then the map $F(t, x) = (t - z(t), x)$*

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