# CONJUGATE LOCI IN GRASSMANN MANIFOLDS

BY YUNG-CHOW WONG

Communicated by S. Smale, September 29, 1967

1. **Introduction.** In the tangent space $M_x$ to a Riemannian manifold $M$ at the point $x$, a conjugate point $v$ is a point at which the differential of the exponential map $\exp_x$: $M_x \to M$ is singular. In $M$, a point $y$ is a conjugate point to $x$ if $y = \exp_x v$ for some conjugate point $v$ in $M_x$. The conjugate locus in $M_x$ is the set of conjugate points in $M_x$, and the conjugate locus in $M$ at $x$ is the set of conjugate points to $x$.

Though there are a number of general results on the conjugate locus either in $M_x$ or in $M$ ([4], [6, p. 59], [11], [12] and [13]), the precise nature of this locus in special Riemannian manifolds seems to be known only in a few cases, such as the sphere, the projective spaces, and some two-dimensional manifolds ([2, pp. 225–226], [9] and [10]). In the present note, we give a complete description of the conjugate locus at a point in the real, complex or quaternionic Grassmann manifolds. Besides being useful and interesting, this information will extend the range of problems recently studied by Klingenberg [8], Allamigeon [1], Green [5] and Warner [12, 13]. The conjugate locus in the tangent space to a Grassmann manifold is more complex and will be the subject of a future note.

In §2, we describe the Schubert varieties of which the conjugate locus in a Grassmann manifold is composed. In §3, we give some results concerning conjugate points in a Grassmann manifold. In §4, we state our main theorem. Details and proof will be omitted. For background information, the reader is referred to the author's paper [14].

2. **Some Schubert varieties** (cf. [3, Chapter 4] and [7, Chapter 14]). Let $F$ be the field $R$ of real numbers, the field $C$ of complex numbers, or the field $H$ of real quaternions; $F^{n+m}$ an $(n+m)$-dimensional left vector space over $F$ provided with a positive definite hermitian inner product; $G_n(F^{n+m})$ the Grassmann manifold of $n$-planes in $F^{n+m}$.

In $F^{n+m}$, let $\boldsymbol{P}$ be a fixed $p$-plane $(1 < p < n+m)$, $\boldsymbol{Z}$ a variable $n$-plane, and

$$V_l = \{\boldsymbol{Z}: \dim(\boldsymbol{Z} \cap \boldsymbol{P}) \geqq l\} \qquad (l \geqq 0),$$

$$W_l = V_l \backslash V_{l+1} = \{\boldsymbol{Z}: \dim(\boldsymbol{Z} \cap \boldsymbol{P}) = l\} \qquad (l \geqq 0).$$