A GAME WITH NO SOLUTION[1]

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1. Introduction. In 1944 von Neumann and Morgenstern [**2**] introduced a theory of solutions for n-person games in characteristic function form. The main mathematical question concerning their model is whether every game has at least one solution. This announcement describes a ten-person game which has no solution. The essential definitions for an n-person game will be reviewed briefly before the particular example is given. The proof that the game has no solution will then be sketched; a detailed proof will be published elsewhere.

2. **Definitions.** An *n-person game* is a pair (N, v) where $N = \{1, 2, \cdots, n\}$ is the set of players and v is a characteristic function on 2^N, i.e., v assigns the real number $v(S)$ to each subset S of N and $v(\phi) = 0$. The set of *imputations* is

$$A = \left\{x: \sum_{i\in N} x_i = v(N) \text{ and } x_i \geqq v(\{i\}) \text{ for all } i \in N\right\}$$

where $x = (x_1, x_2, \cdots, x_n)$ is a vector with real components. For any $X \subset A$ and nonempty $S \subset N$, define $\text{Dom}_S X$ to be the set of all $x \in A$ such that there exists a $y \in X$ with $y_i > x_i$ for all $i \in S$ and with $\sum_{i\in S} y_i \leqq v(S)$. Let $\text{Dom}\, X = \bigcup_{S\subset N} \text{Dom}_S X$. Also let $\text{Dom}^{-1} X$ be the set of all $y \in A$ such that there exists $x \in X$ with $x \in \text{Dom}\{y\}$. A subset K of A is a *solution* if $K \cap \text{Dom}\, K = \phi$ and $K \cup \text{Dom}\, K = A$. If $X \subset A$ and $K' \subset X$, then K' is a *solution for X* if $K' \cap \text{Dom}\, K' = \phi$ and $K' \cup \text{Dom}\, K' \supset X$. The *core* of a game is

$$C = \left\{x \in A: \sum_{i\in S} x_i \geqq v(S) \text{ for all } S \subset N\right\}.$$

For any solution K, $C \subset K$ and $K \cap \text{Dom}\, C = \phi$.

A characteristic function v is *superadditive* if $v(S_1 \cup S_2) \geqq v(S_1) + v(S_2)$ whenever $S_1 \cap S_2 = \phi$. The game listed below does not have a superadditive v as assumed in the classical theory. However, it is equivalent solutionwise to a game with a superadditive v. (See Gillies [**1**, p. 68].)

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