STRONGLY CONVEX METRICS IN CELLS[1]

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The following question was raised by Bing in [2]: "If an n-dimensional compact topological space has a metric which is strongly convex and without ramifications (defined below), is it necessarily homeomorphic to the Euclidean n-cell?" Lelek and Nitka [5] answered this affirmatively for $n \leqq 2$; we outline below a proof that the answer is also yes when $n=3$. Although the question remains open in higher dimensions, we also give an affirmative answer when the space is assumed to be a manifold (=manifold with boundary) and $n \neq 4$ or 5. In fact with this further assumption we may omit the "without ramifications" requirement when $n \leqq 3$.

If X is a space and $x, y, m \in X$, then m is called a *midpoint* of x and y (with respect to a metric d on X) if $d(x, m) = d(m, y) = \frac{1}{2}d(x, y)$. The metric is *strongly convex* (SC) if each pair of points has a unique midpoint and *without ramifications* (WR) if no midpoint of x and y is a midpoint of x' and y unless $x' = x$. Both of these properties are enjoyed by the usual metric on Euclidean spaces and cells, and they are preserved under cartesian products in the following sense:

PROPOSITION 1. *If d_i is a* SC (*or* WR) *metric on X_i, $i = 1, \cdots, n$ then $d(x, y) = \sum [d_i(x_i, y_i)^2]^{1/2}$ determines a* SC (*resp.* WR) *metric on $X = X_1 \times \cdots \times X_n$. (Here x_i denotes the* ith *coordinate of $x = \{x_i\} \in X$ and the sum extends over $i = 1, \cdots, n$.)*

Indeed an easy exercise in inequalities verifies that $\{m_i\}$ is a midpoint of $\{x_i\}$ and $\{y_i\}$ in (X, d) iff each m_i is a midpoint of x_i and y_i in (X_i, d_i).

Strongly convex metrics. Joining any two points in a complete SC metric space, there is a unique arc (called a *segment*) which is isometric to a closed interval of the real line [7]. It follows that the intersection of any two segments is connected or empty. In a compact SC metric space, segments vary continuously with their endpoints, allowing one to imitate some of the tricks available in Euclidean space. For example, by moving points along segments toward a fixed basepoint we can obtain deformations of the space and prove (see [2])

[1] These results are a portion of the author's Ph.D. thesis, written under Joseph Martin at the University of Wisconsin.