ON THE COMMENSURABILITY CLASS OF THE SIEGEL MODULAR GROUP

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The purpose of this note is to determine the commensurability class of $G_{\mathfrak{o}}$, where G is either the Sympletic Group Sp_n or the Special Linear Group Sl_n, and $\mathfrak{o}$ is the ring of integers of a number field k of finite degree over the rationals. This problem has been solved in the local case by Hijikata [6], and Bruhat [3], for Sl_n and by Allan [1] and Bruhat-Tits [8] for Sp_n. The only known global solution is due to Helling [5], in the case of Sl_2. We shall exhibit a countable family of arithmetic groups in G, such that every maximal arithmetic group is conjugate to one group of the family, and if $\mathfrak{o}$ is a principal ideal ring, every group of this family is a maximal arithmetic group.

We know that if G is a connected semisimple linear group defined over k, which is absolutely irreducible as a matrix group, then a maximal arithmetic group Δ is the normalizer of its intersection Δ' with G_k, and Δ' is selfnormalizer in G_k. Our problem is then to determine the selfnormalizer subgroups of G_k whose normalizer is maximal, because the maximal groups which are the normalizer of maximal subgroups of G_k has already been determined in [1], for the groups we are interested in.

Let $\mathfrak{p}$ be a finite prime spot of k and $k(\mathfrak{p})$ be the completion of k at $\mathfrak{p}$; let $\mathfrak{o}(\mathfrak{p})$ be the ring of integers of $k(\mathfrak{p})$. Let Δ be an arithmetic group contained in G_k and let $\Delta^{\mathfrak{p}}$ be its $\mathfrak{p}$-adic closure in $G_{k(\mathfrak{p})}$. Assume that $N_k(\Delta)=\Delta$, and let $\mathfrak{J}$ be the ideal in $\mathfrak{o}$ generated by all $\mu\in\mathfrak{o}$ such that μg has only algebraic integral entries for all $g\in N(\Delta)$. Let $\Omega(\mathfrak{p})$ be the algebraic closure of $k(\mathfrak{p})$. It is easy to see that $N(\Delta)$ can be imbedded in $N(\Delta^{\mathfrak{p}})$. Clearly the groups Δ which we want to determine are such that $\Delta=\bigcap(G_k\cap\Delta^{\mathfrak{p}})$, the intersection taken over all finite primes of k. From now on we shall assume that all groups Δ considered, have this property. We shall also assume that G has the strong approximation property (see [7]). Then it is very simple to verify the following two lemmas:

LEMMA 1. *$N_k(\Delta)=\Delta$ if and only if $N_{k(\mathfrak{p})}(\Delta^{\mathfrak{p}})=\Delta^{\mathfrak{p}}$ for all $\mathfrak{p}$. Similarly Δ is maximal in G_k if and only if $\Delta^{\mathfrak{p}}$ is maximal in $G_{k(\mathfrak{p})}$ for all $\mathfrak{p}$.*

LEMMA 2. *Δ is conjugate to Γ in G_k if and only if $(\Delta)^{\mathfrak{p}}$ is conjugate to $(\Gamma)^{\mathfrak{p}}$ in $G_{k(\mathfrak{p})}$ for all $\mathfrak{p}$.*