HOMEOMORPHISMS OF $S^n \times S^1$

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It is the object of this note to describe several results about homeomorphisms of $S^n \times S^1$. The main tool is Theorem 2: Every homeomorphism of $S^n \times S^1$ extends to a homeomorphism of $D^{n+1} \times S^1$. The proof is sketched in §1 and the result used in §2 to yield information about deformations of homeomorphisms. §3 contains results on the division of S^{n+2} by $S^n \times S^1$.

1. DEFINITION. Two submanifolds L^{n-1} and M^{n-1} in N^n are said to be *transverse* if there is a coordinate system about each point of $L^{n-1} \cap M^{n-1}$ in which L^{n-1} and M^{n-1} look like intersecting hyperplanes in R^n.

THEOREM 1. *If Σ is a locally flat n-sphere in $S^n \times S^1$, $n > 1$, then Σ bounds a locally flat $(n+1)$-disk Δ in $D^{n+1} \times S^1$ which is transverse to $S^n \times S^1$.*

PROOF (SKETCH). If Σ bounds a disk in $S^n \times S^1$, the proof is trivial, so assume that it does not. Look at the universal covering space $S^n \times R^1$ of $S^n \times S^1$ with covering translation T, and let Σ_0 be a lifting of Σ to $S^n \times R^1$. Since T is stable, the region between Σ_0 and $T\Sigma_0$ is an annulus (Brown and Gluck [1]). Thus there is a homeomorphism $g: S^n \times R^1 \rightarrow S^n \times R^1$ such that $Tg = gT$ and $g(S^n \times \{0\}) = \Sigma_0$. It will be sufficient to construct a disk $\Delta_0 \subset D^{n+1} \times R^1$ such that

(1) Δ_0 is locally flat,

(2) Δ_0 is transverse to $S^n \times R^1$ along Σ_0, and

(3) Δ_0 is disjoint from its translates $T^k\Delta_0$.

Then Δ_0 will project onto the desired Δ.

CONSTRUCTION OF Δ_0. Choose a number M such that $\Sigma_0 \subset S^n \times (-M, M)$. Let A be the annular region on $S^n \times S^1$ between Σ_0 and $S^n \times \{M\}$, and B the disk $D^{n+1} \times \{M\}$. Then $A \cup B$ is a locally flat manifold, which is a disk by the generalized Shoenflies theorem (Brown [2], [3]).

Give R^{n+1} polar coordinates $(r, x) \rightarrow rx$ where $r \in [0, \infty)$ and $x \in S^n$.

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