ITERATED PATH INTEGRALS AND GENERALIZED PATHS[1]

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Let $\mathfrak{M}$ be a C^∞ manifold with a countable basis. For convenience, it is assumed that $\mathfrak{M}$ is Riemannian. Let $\mathfrak{P}$ be the set of "reduced" piecewise C^1 paths having a common initial point p in $\mathfrak{M}$ such that each $\alpha \in \mathfrak{P}$ is parameterized by arc length. By a reduced path α: $[0, l] \to \mathfrak{M}$, we mean one such that there exists no $t \in (0, l)$ with $\alpha(t-s) = \alpha(t+s)$ for $|s|$ sufficiently small.

Let Ω be the vector space (over the real number field R) of C^∞ 1-forms on $\mathfrak{M}$. Elements of Ω will be denoted by $w, w_1, w_2, \cdots$. Let α^t be the restriction $\alpha | [0, t]$, $0 \leqq t \leqq l$. Let $\int_\alpha w_1$ be the usual integral, and define, for $r > 1$,

$$\int_\alpha w_1 \cdots w_r = \int_0^l \left(\int_{\alpha^t} w_1 \cdots w_{r-1} \right) w_r(\alpha(t), \dot{\alpha}(t)) dt.$$

Each iterated integral $\int w_1 \cdots w_r$ is thus a real valued function on $\mathfrak{P}$. The totality of iterated integrals together with the constant functions on $\mathfrak{P}$ generates a subalgebra F of the R-algebra of real valued functions on $\mathfrak{P}$. The R-algebra F is of interest for two reasons: (a) Elements of F have geometrical significance of the manifold $\mathfrak{M}$. (b) It follows from results in [1] that F contains sufficiently many functions on $\mathfrak{P}$ as to separate the points of $\mathfrak{P}$.

The purpose of this note is to give some indication of the structure of F. In particular, Theorem 2 implies that, if $\mathfrak{M} = R^n$, then F contains a dense subalgebra which is algebraically isomorphic with a polynomial algebra of, at most, countably many indeterminates.

We shall also introduce the notion of a generalized path in $\mathfrak{M}$ which is obtained through a process of dualization in a manner somewhat more complicated than that of a 1-dimensional current. (See [4].) The multiplication of generalized paths is nonabelian.

A detailed account will be given in a forthcoming paper.

1. Given any compact subset K of $\mathfrak{M}$, define the seminorm $\| \ \|_K$ of Ω such that

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