PROJECTIONS ONTO SEPARABLE C^*-SUBALGEBRAS OF A W^*-ALGEBRA

BY CHARLES A. AKEMANN

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This paper generalizes a result of Grothendieck [2] to the case of a nonabelian W^*-algebra.

THEOREM. *Let M be a W^*-algebra and N a C^*-subalgebra of M which is separable in the norm topology. Then there exists a bounded projection of M onto N iff N is finite-dimensional.*

PROOF. Suppose $P: M \to N$ is a bounded projection. If N is infinite-dimensional, by [3] it must have an infinite-dimensional abelian *subalgebra. Call it N_0. Then N_0 is isomorphic to $C_0(X)$, the continuous functions vanishing at infinity on some locally compact Hausdorff space X. Thus, by Urysohn's Lemma, there exists a sequence $\{b_k\}$ of orthogonal, positive elements of N_0 with $\|b_k\| = 1$ for all $k = 1, 2, \cdots$.

By the Hahn-Banach Theorem we may choose $\{f_k\}$ in N^* with $\|f_k\| = 1$ and $f_k(b_2) = \delta_{kj}$. Since N is separable, the unit ball B of N^* is weak* sequentially compact. Thus, taking subsequences if necessary, we may assume that $\{f_k\}$ is weak* convergent.

Now $P^*: N^* \to M^*$ is weak* continuous, so $\{P^*(f_k)\}$ is weak* convergent in M^*. Also $P^*(f_k)(b_j) = f_k(b_j) = \delta_{kj}$. But $\sum_{k=1}^{\infty}(b_k)$ exists in the s-topology of M, so Theorem III.7 of [1] applies to give $\sum_{k=1}^{\infty} f_j(b_k)$ exists uniformly for $j = 1, 2, \cdots$, a contradiction. Thus N is finite dimensional.

The reverse implication is well known. Q.E.D.

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UNIVERSITY OF PENNSYLVANIA