SIMILARITY FOR SEQUENCES OF PROJECTIONS

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We consider sequences $\{P_n\}_{n=0,1,\cdots}$ of (not necessarily selfadjoint) projections in a Hilbert space H satisfying the orthogonality conditions $P_nP_m=\delta_{mn}P_n$. For brevity, such a sequence $\{P_n\}$ will be called a *p-sequence.* A p-sequence $\{E_n\}$ is *selfadjoint* if $E_n^*=E_n$ for all n. A selfadjoint p-sequence $\{E_n\}$ is *complete* if $\sum E_n$, which always converges strongly, is equal to the identity.

The object of this note is to prove the following theorem.

THEOREM. *Let $\{P_n\}$ be a p-sequence, and $\{E_n\}$ a complete selfadjoint p-sequence. Furthermore, assume that*

$$\dim P_0 = \dim E_0 = m < \infty, \tag{1}$$

$$\sum_{n=1}^{\infty} \|E_n(P_n - E_n)u\|^2 \leqq c^2\|u\|^2 \quad \text{for all } u \in H, \tag{2}$$

where c is a constant such that $0 \leqq c < 1$. Then $\{P_n\}$ is similar to $\{E_n\}$, that is, there exists a nonsingular linear operator W such that

$$P_n = W^{-1}E_nW, \qquad n = 0, 1, 2, \cdots. \tag{3}$$

PROOF. First we shall show that

$$W = \sum_{n=0}^{\infty} E_nP_n \tag{4}$$

exists in the strong sense. Since $\sum E_n=1$ strongly, it suffices to show that $\sum(E_n-E_nP_n)=\sum E_n(E_n-P_n)$ converges strongly. But this is true since

$$\left\|\sum_{n=m}^{m+p} E_n(E_n - P_n)u\right\|^2 = \sum_{n=m}^{m+p} \|E_n(E_n - P_n)u\|^2 \to 0, \qquad m \to \infty, \tag{5}$$

by (2). Incidentally, we note that (5) implies $\|A\| \leqq c < 1$, where

$$A = \sum_{n=1}^{\infty} E_n(E_n - P_n) = 1 - E_0 - \sum_{n=1}^{\infty} E_nP_n. \tag{6}$$

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[1] This work represents part of the results obtained while the author held a Miller Research Professorship.