# NONLINEAR MAPPINGS OF NONEXPANSIVE AND ACCRETIVE TYPE IN BANACH SPACES

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**Introduction.** Let $X$ be a Banach space. If $T$ and $U$ are mappings (in general nonlinear) with domains $D(T)$ and $D(U)$ in $X$ and with values in $X$, then $U$ is said to be *nonexpansive* if for all $u$ and $v$ in $D(U)$,

$$\|U(u) - U(v)\| \leqq \|u - v\|,$$

while $T$ is said to be *accretive* if for all $u$ and $v$ of $D(T)$,

$$(T(u) - T(v), w) \geqq 0, \quad w \in J(u - v),$$

where $(x, w)$ denotes the pairing of the element $x$ of $X$ and the element $w$ of the conjugate space $X^*$ and for each $x$ in $X$, $J(x)$ is the convex subset of $X^*$ given by

$$J(x) = \{w \mid w \in X^*, (w, x) = \|x\|^2, \|w\| = \|x\|\}.$$

There are two important connections between the classes of nonexpansive and of accretive mappings which give rise to a strong connection between the fixed point theory of nonexpansive mappings and the mapping theory of accretive maps. These are:

(1) If $U$ is a nonexpansive mapping of $D(U)$ into $X$, and if we set $T=I-U$, $D(T)=D(U)$, then $T$ is an accretive mapping of $D(T)$ into $X$.

(2) If $\{U(t), t\geqq 0\}$ is a semigroup of (nonlinear) mappings of $X$ into $X$ with infinitesimal generator $T$, then all the mappings $U(t)$ are nonexpansive if and only if $(-T)$ is accretive.

For the special case when $X$ is a Hilbert space $H$ (and the concept of an accretive mapping coincides with that of a *monotone* mapping), the writer in Browder [**3**], [**4**] used the observation (1) above and the theory of monotone mappings in Hilbert space to prove the following fixed point theorem for nonexpansive maps: *If $C$ is a closed bounded convex subset of the Hilbert space $H$, $U$ a nonexpansive mapping of $C$ into $C$ which maps the boundary of $C$ into $C$, then $U$ has a fixed point in $C$.* This line of argument has also been exploited to yield further results on the existence and calculation of fixed points of nonexpansive mappings in Hilbert space and in the class of Banach spaces having weakly continuous duality mappings (like the spaces $l^p$,