STRONG HOMOTOPY EQUIVALENCE OF 3-MANIFOLDS

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1. **Introduction.** Let M be a topological space and let X be a compact subset of M. After [2], we say that X has property UV^∞ (or "$X \in UV^\infty$") in M if for each open set $U \subset M$ such that $X \subset U$, there is an open set V such that $X \subset V \subset U$ and V is contractible to a point in U. It is known [2] that each finite-dimensional compact absolute retract (i.e., retract of a cell) has property UV^∞ under some embedding in Euclidean space, and that if one embedding of a compact set X in a manifold has property UV^∞, then so does every embedding of X in a manifold.

Armentrout has shown (§10 of [2]) that if M^n and N^n are *closed* (i.e., compact and boundaryless) topological n-manifolds, and if f is a continuous mapping of M^n onto N^n such that $f^{-1}(y) \in UV^\infty$ for each $y \in N^n$, then f is a homotopy equivalence. We shall call such a mapping a *strong homotopy equivalence of* M^n onto N^n. If $n=3$ and if f is *cellular* (i.e., each set $f^{-1}(y)$ is cellular—and hence UV^∞), then he has shown [1] that M^3 and N^3 are homeomorphic. It is our purpose here to note that if there is a strong homotopy equivalence of M^3 onto N^3, then M^3 and N^3 differ by only a finite number of homotopy 3-cells (Corollary 2.1). Hence, modulo the Poincaré conjecture, M^3 and N^3 are homeomorphic. If there is also a strong homotopy equivalence of N^3 onto M^3, then M^3 and N^3 *are* homeomorphic (independently of the Poincaré conjecture).

If X is a compact subset of the interior of a piecewise-linear n-manifold M^n, $n \geqq 3$, and if $X \in UV^\infty$, then we shall say that X satisfies the *cellularity criterion* in M^n if for each open set $U \subset M^n$ such that $X \subset U$, there is an open set V such that $X \subset V \subset U$ and each loop in $V-X$ is contractible in $U-X$. If $n \geqq 5$ and $X \in UV^\infty$, then X is cellular (with respect to piecewise-linear cells) in M^n if and only if the cellularity criterion holds (see [5]). For the situation in the 3-dimensional case, see Theorem 1.

We shall use E^n and S^n to denote Euclidean n-space, and the n-sphere, respectively. The term "manifold" applies only to a connected space, unless stated otherwise. If G is a disjoint collection of closed subsets of a space X such that the union of the elements of G is

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