CONSTRUCTIVE PROOF OF THE EXISTENCE OF MULTIPLICATIVE FUNCTIONALS IN COMMUTATIVE SEPARABLE BANACH ALGEBRAS

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Gelfand's 1941 proof of the existence of multiplicative functionals in commutative Banach algebras is essentially based on Zorn's axiom.

In 1961, P. J. Cohen [3] gave a constructive (i.e. free from Zorn's axiom) way to get rid of Banach algebras in some of their applications.

This year, E. Bishop [1], [2] has presented a theory of Banach algebras in the frame of L. E. J. Brouwer's constructivist ideas. Therefrom it is easy to deduce a constructive proof of the existence of multiplicative functionals. However this proof would be needlessly intricate when just interested in constructive methods.

Here is a simple constructive proof of Gelfand's theorem.

1. Let A be a commutative separable Banach algebra with unit 1 throughout the paper.

Let us recall some properties of ideals of A.

(a) *0, A, $\sum_{i=1}^{m} x_iA$ and $\mathfrak{J}+\sum_{i=1}^{m} x_iA$ are ideals of A whenever $x_1, \cdots, x_m \in A$ and $\mathfrak{J}$ is an ideal of A.*

(b) *If an ideal $\mathfrak{J}$ contains an invertible element, then $\mathfrak{J}=A$.*

(c) *Let $\mathfrak{J}\neq A$ be an ideal, then $d[1, \mathfrak{J}]=1$.*

Since $0\in\mathfrak{J}$, $d[1, \mathfrak{J}]\leqq 1$. Moreover if $d[1, \mathfrak{J}]<1$, there exists $x_0\in\mathfrak{J}$ such that $d[1, x_0]<1$. Then x_0^{-1} exists and consequently $1=x_0x_0^{-1}$ belongs to $\mathfrak{J}$.

(d) *Let $\mathfrak{J}\neq A$ be an ideal. If $1-xy\in\mathfrak{J}$, then $d[x, \mathfrak{J}]\geqq 1/\|y\|$.*

In fact, $\mathfrak{J}\neq A$ implies $d[1, \mathfrak{J}]=1$ and since $1-xy\in\mathfrak{J}$, we have $d[xy, \mathfrak{J}]=1$ and $d[xy, \mathfrak{J}]\leqq d[xy, y\mathfrak{J}]\leqq\|y\|d[x, \mathfrak{J}]$.

2. We need a lemma, which is a direct version of the classical fact that the spectrum of the Banach algebra E/A is not void.

Let $\mathfrak{J}\neq A$ be an ideal. Then for all $x\in A$, there exists $z\in\mathbf{C}$ such that $\mathfrak{J}+(x-z)A\neq A$.

Suppose there exists an ideal $\mathfrak{J}\neq A$ and $x\in A$ such that $\mathfrak{J}+(x-z)A=A$ for all $z\in\mathbf{C}$.

Then for all $z\in\mathbf{C}$, there is at least one element $a(z)\in A$ with $1-(x-z)a(z)\in\mathfrak{J}$.

Let $\mathfrak{x}$ be any continuous linear functional in A vanishing on $\mathfrak{J}$.

(a) *$\mathfrak{x}[a(z)]$ depends only on $z\in\mathbf{C}$ and not on the choice of $a(z)$.*